Continuity of Polynomials, Trigonometric Functions and Exponentials

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Continuity of Polynomial Functions

Recall: We showed that if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

then $\lim_{x \to a} p(x) = p(a)$.

Theorem: [Continuity of Polynomials]

Let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then $p(x)$ is continuous at each $a \in \mathbb{R}$ with $a > 0$.

Continuity of $sin(x)$ and $\cos(x)$

Recall: We have seen that

$$\lim_{x \to 0} \sin(x) = 0 = \sin(0) \quad \text{and} \quad \lim_{x \to 0} \cos(x) = 1 = \cos(0).$$

That is, both $\sin(x)$ and $\cos(x)$ are continuous at $a = 0$.

Theorem: [Continuity of $sin(x)$ and $\cos(x)$]

Both $\sin(x)$ and $\cos(x)$ are continuous at each $a \in \mathbb{R}$.

Proof: Observe that

$$\begin{aligned}
\lim_{x \to a} \sin(x) &= \lim_{h \to 0} \sin(a + h) \\
&= \lim_{h \to 0} \sin(a)\cos(h) + \sin(h)\cos(a) \\
&= \sin(a) \cdot 1 + 0 \cdot \cos(a) \\
&= \sin(a)
\end{aligned}$$

Continuity of $sin(x)$ and $\cos(x)$

Proof (continued): Next observe that

$$\begin{aligned}
\lim_{x\to a} \cos(x) &= \lim_{h\to 0} \cos(a+h) \\
&= \lim_{h\to 0} \cos(a)\cos(h) - \sin(a)\sin(h) \\
&= \cos(a)\cdot 1 - \sin(a)\cdot 0 \\
&= \cos(a)
\end{aligned}$$

Continuity of e^x

Note:

1) It is actually not an easy task to prove the continuity of the function $f(x) = e^x$.
2) The easiest way to show that e^x is continuous is to realize that it can be defined by a special type of series known as a **power series**.
3) If e^x is continuous at $x = 0$, then it is continuous everywhere.

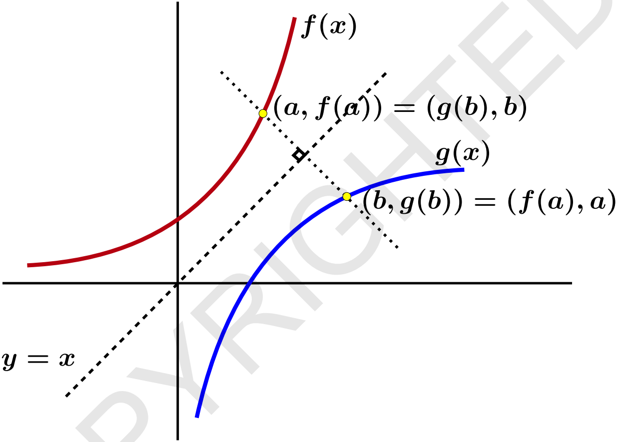
First observe that if e^x is continuous at $x = 0$, then

$$1 = e^0 = \lim_{h \to 0} e^h.$$

Hence

$$\begin{aligned} \lim_{x \to a} e^x &= \lim_{h \to 0} e^{a+h} \\ &= \lim_{h \to 0} e^a e^h \\ &= e^a \lim_{h \to 0} e^h = e^a. \end{aligned}$$

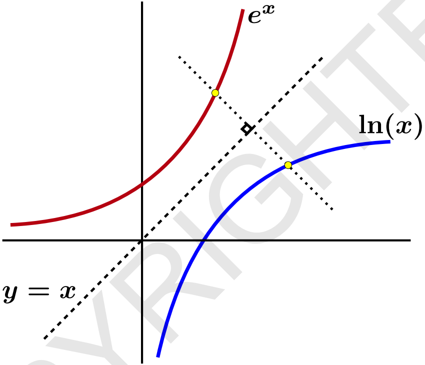
Continuity of Inverses



Theorem: [Continuity of Inverses]

Assume that $f(x)$ is invertible with inverse $g(y)$. If $f(a) = b$ and if $f(x)$ is continuous at $x = a$, then $g(y)$ is continuous at $y = b = f(a)$.

Continuity of e^x and $\ln(x)$



Theorem: [Continuity of e^x and $ln(x)$]

We have

1) e^x is continuous at each $a \in \mathbb{R}$.
2) $\ln(x)$ is continuous for each $a \in \mathbb{R}$ with $a > 0$.