One-sided Limits

Created by

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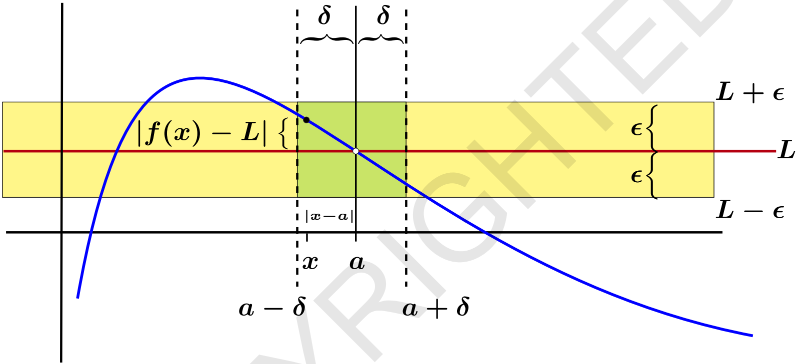
Formal Definition of a Limit



Recall: Formal Definition of a Limit

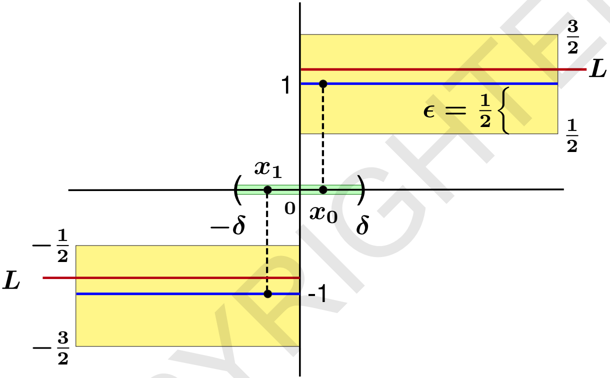
We say that L is the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 <| x - a |< \delta,$$

then

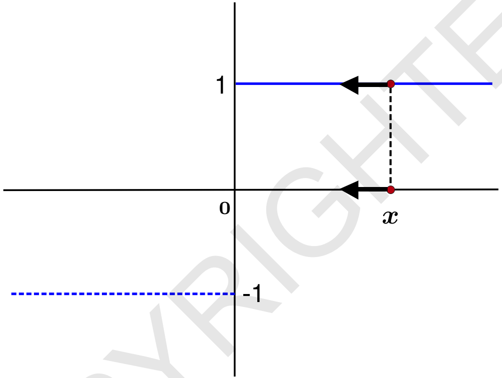
$$| f(x) - L |< \epsilon.$$

Example



Example: Let $f(x) = \frac{|x|}{x}$. $\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

Example



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If $x > 0$ and $x \to 0$, then $f(x) \to 1$.

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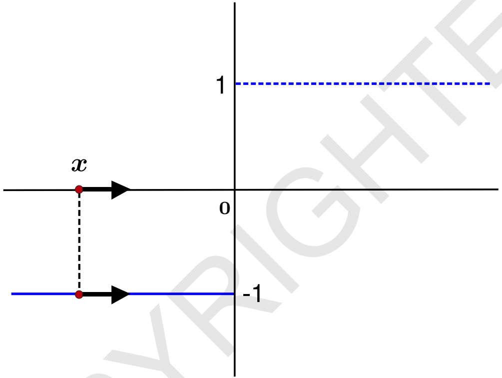
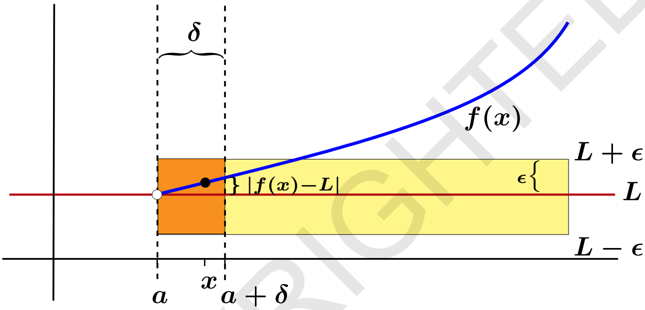


Example: Let $f(x) = \frac{|x|}{x}$. $\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

If $x > 0$ and $x \to 0$, then $f(x) \to 1$.

If $x < 0$ and $x \to 0$, then $f(x) \to -1$.

One-sided Limits



Definition: [Limit from Above]

We say that L is the limit of $f(x)$ as x approaches a from above (or from the right), if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $a < x < a + \delta$, then

$$| f(x) - L | < \epsilon.$$

We write $\lim_{x \to a^+} f(x) = L$.

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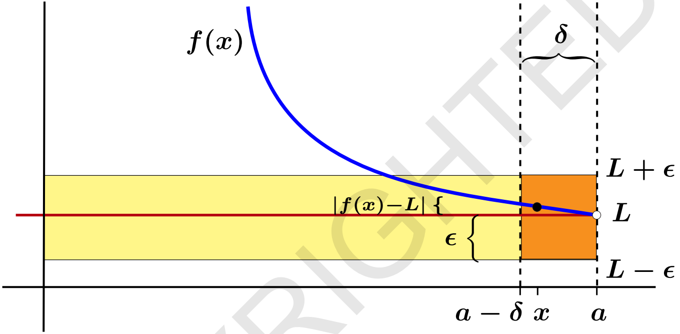


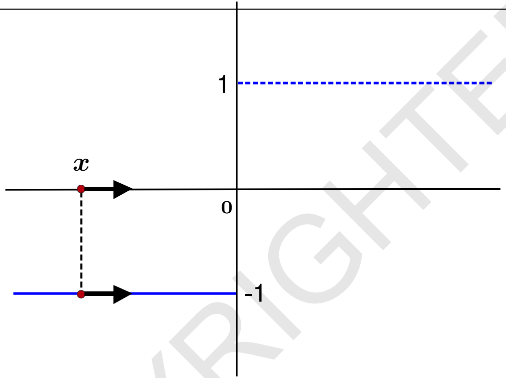
Definition: [Limit from Below]

We say that L is the limit of $f(x)$ as x approaches a from below (or from the left), if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $a - \delta < x < a$, then

$$| f(x) - L | < \epsilon.$$

We write $\lim_{x \to a^-} f(x) = L$.

Example



Example: We know that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. However,

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1$$

and

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

One-sided Limits vs. Limits

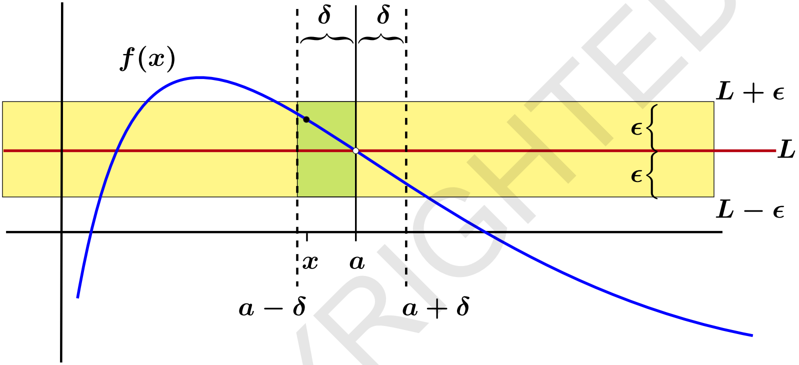
Theorem

The following are equivalent:

1. $\lim_{x \to a} f(x) = L$.
2. Both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist with

$$\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).$$

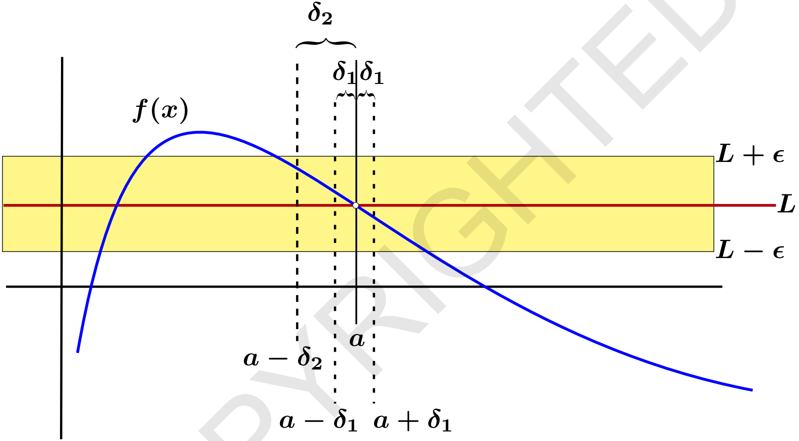
One-sided Limits



Assume that $\lim_{x \to a} f(x) = L$

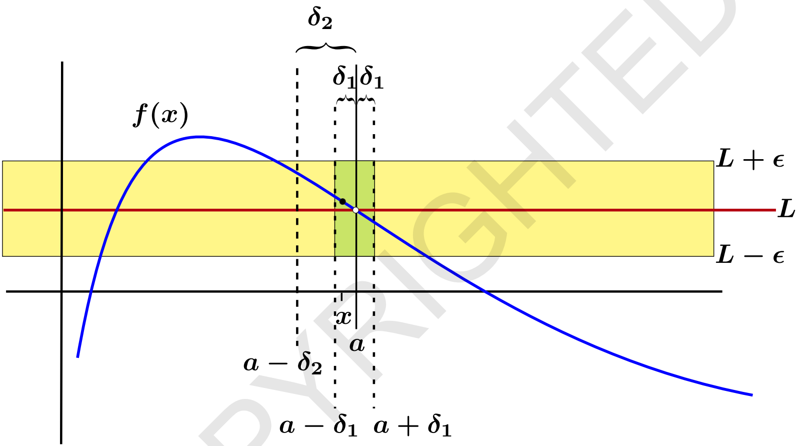
$$\Rightarrow \quad \lim_{x \to a^+} f(x) = L, \text{ and}$$
$$\lim_{x \to a^-} f(x) = L.$$

One-sided Limits



Assume that $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$

One-sided Limits



Assume that $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \Rightarrow \lim_{x \to a} f(x) = L$.