Limits of Functions: Part II

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Formal Definition of a Limit of a Function

Definition: [Limit of a Function]

We say that L is the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 <\mid x - a \mid< \delta,$$

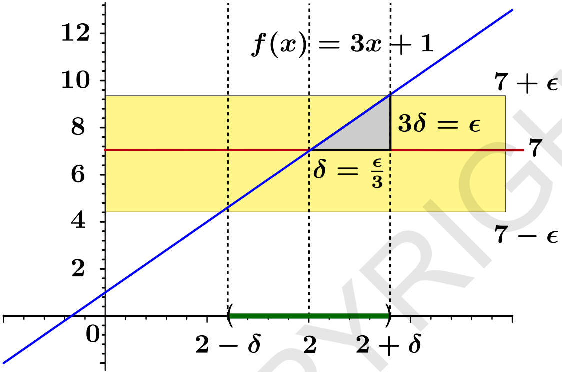
then

$$\mid f(x) - L \mid< \epsilon.$$

We write

$$\lim_{x \to a} f(x) = L.$$

Proving the Limit Exists



Example: Show that

$$\lim_{x \to 2} 3x + 1 = 7.$$

$$\frac{\epsilon}{\delta} = \frac{\text{Rise}}{\text{Run}} = 3$$

$$\Rightarrow \delta = \frac{\epsilon}{3}.$$

Algebraically, we want

$$\begin{aligned} |\,(3x+1) - 7\,| < \epsilon \quad &\Leftrightarrow \quad |\,3x - 6\,| < \epsilon \\ &\Leftrightarrow \quad 3\,|\,x - 2\,| < \epsilon \\ &\Leftrightarrow \quad |\,x - 2\,| < \tfrac{\epsilon}{3} \end{aligned}$$

So if $\delta = \frac{\epsilon}{3}$ and $0 <|\, x - 2 \,|< \delta$, we have $|\,(3x + 1) - 7\,|< \epsilon$.

Example

Remark: If $f(x) = mx + b$ where $m \neq 0$, then

$$\lim_{x \to a} mx + b = m(a) + b.$$

Given $\epsilon > 0$, if

$$\delta = \frac{\epsilon}{\mid m \mid}$$

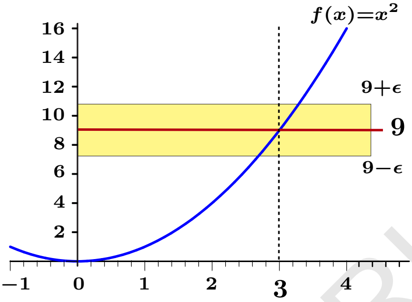
and if $0 <\mid x - a \mid< \delta$, then

$$\begin{aligned} \mid f(x) - (m(a) + b) \mid &= |(mx + b) - (ma + b)| \\ &= |m| \cdot |x - a| \\ &< |m| \cdot \frac{\epsilon}{\mid m \mid} \\ &= \epsilon \end{aligned}$$

Example:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2.$$

Example



Example: Show that

$$\lim_{x\to 3} x^2 = 9.$$

Let $\epsilon > 0$. We want

$$0 <\mid x - 3 \mid< \delta$$

to imply

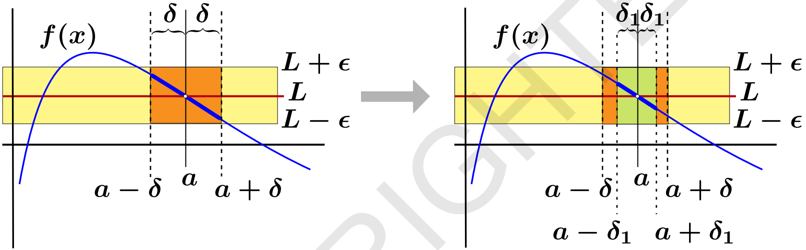
$$\mid x^2 - 9 \mid = \mid x - 3 \mid\mid x + 3 \mid < \epsilon.$$

We might choose $\delta = \frac{\epsilon}{|x+3|}$ since if $0 < |x - 3| < \frac{\epsilon}{|x+3|}$, then

$$\mid x^2 - 9 \mid< \frac{\epsilon}{|x+3|} \cdot |x+3| = \epsilon.$$

Note: $\frac{\epsilon}{|x+3|}$ is not a constant!

An Important Observation



$f(x) \in (L-\epsilon, L+\epsilon)$ $\qquad$ $\delta_1 < \delta$ and $f(x) \in (L-\epsilon, L+\epsilon)$

Observation: In the definition of a limit, if we find a δ that works for a particular ϵ, then **any smaller** δ will also satisfy the definition of the limit of a function for the same ϵ.

Trick: In showing that $\lim\limits_{x\to 3} x^2 = 9$, we can always assume that $\delta \leq 1$.

Example (continued)

If

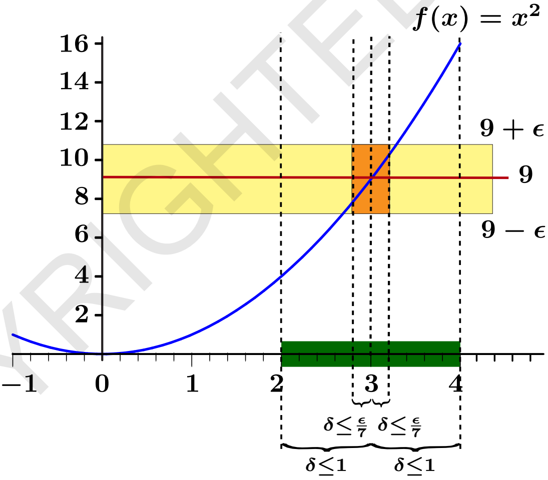
$$0 < |x - 3| < \delta \leq 1,$$

then

$$2 < x < 4$$

so

$$|x + 3| < |4 + 3| = 7.$$



If $\delta < \min(1, \frac{\epsilon}{7})$, then

$$0 < |x - 3| < \delta \Rightarrow$$

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 < \frac{\epsilon}{7} \cdot 7 = \epsilon.$$