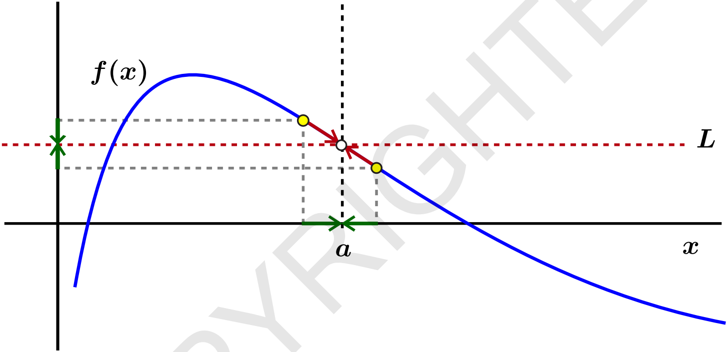
Limits of Functions: Part I

Created by

Barbara Forrest and Brian Forrest

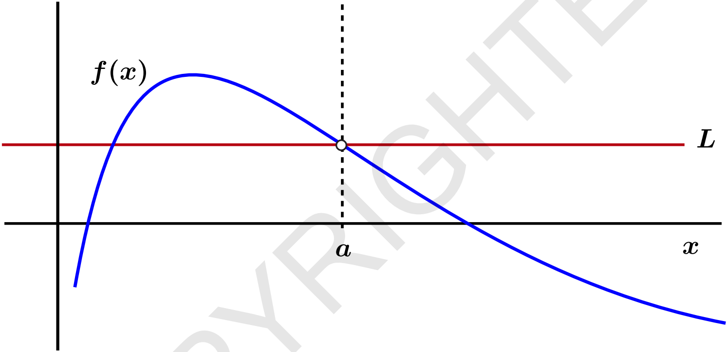
What is a Limit?



Heuristic Definition:

We say that L is the limit of a function $f(x)$ as x approaches a if as x gets closer and closer to a, without ever reaching a, $f(x)$ gets closer and closer to L.

What is a Limit?



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What is a Limit?

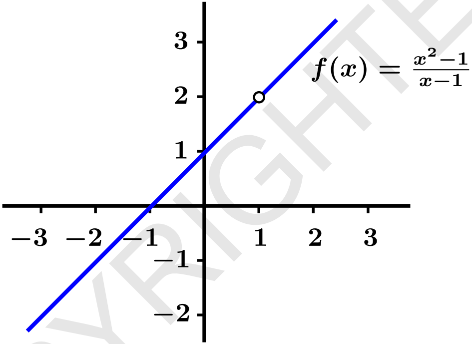
Example: Consider the two functions $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$. It might be tempting to use some algebra and write

$$\begin{aligned} f(x) &= \frac{x^2-1}{x-1} \\ &= \frac{(x+1)(x-1)}{x-1} \\ &= x+1 \\ &= g(x) \end{aligned}$$

Question: Does this mean that $f(x)$ and $g(x)$ are actually the same function?

Answer: *Almost*, but not quite. They have different domains since $f(x)$ is not defined at $x = 1$!

What is a Limit?



Note: The graph of $g(x) = x + 1$ is a straight line with slope 1.

Question: What happens if we graph $f(x) = \frac{x^2-1}{x-1}$?

Answer: The graph of $f(x)$ is the same graph as $g(x) = x + 1$, except there is a *hole* in the graph corresponding to where $x = 1$.

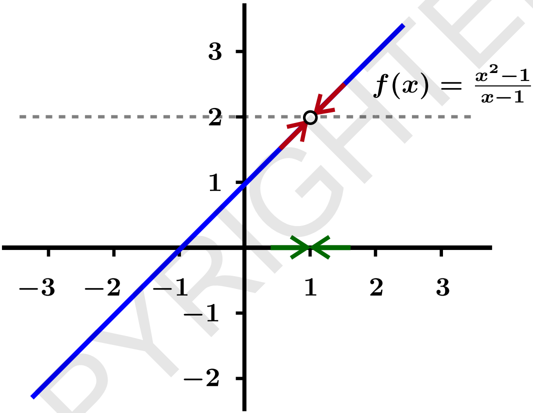
What is a Limit?

We want to focus on the values of $f(x) = \frac{x^2-1}{x-1}$ when x is very close to but not equal to 1. The following is a table of some select values with x near 1.

x	$f(x)$
0	1
0.1	1.1
0.5	1.5
0.75	1.75
0.9	1.9
0.99	1.99
0.999	1.999
0.99999	1.99999
0.99999999	1.99999999

x	$f(x)$
2	3
1.9	2.9
1.5	2.5
1.25	2.25
1.1	2.1
1.01	2.01
1.001	2.001
1.00001	2.00001
1.00000001	2.00000001

What is a Limit?



We can see that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2.

We would like to say that 2 is the limit of $f(x)$ as x approaches 1.

Formal Definition of a Limit

A more robust definition is required.

Improved Heuristic Definition:
L is the limit of $f(x)$ as x approaches a if for any positive tolerance $\epsilon > 0$, we can ensure that $f(x)$ approximates L with error less than ϵ at any x, other than possibly at a itself, provided that x is close enough to a.

Definition: [Limit of a Function]

We say that L is the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

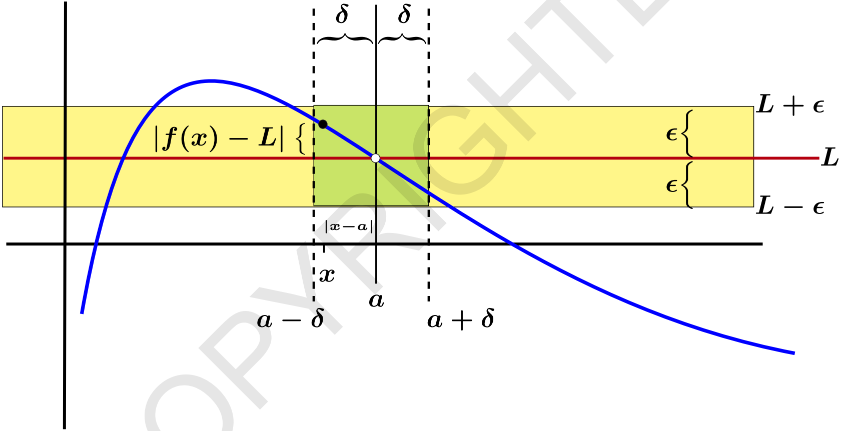
then

$$|f(x) - L| < \epsilon.$$

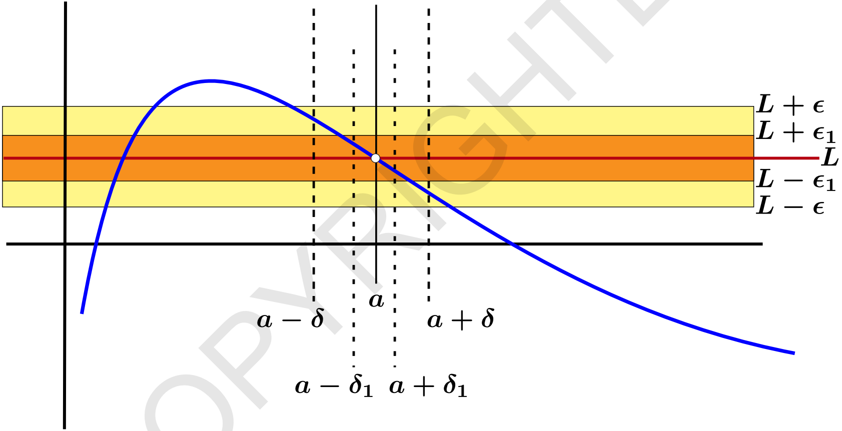
We write

$$\lim_{x \to a} f(x) = L.$$

Formal Definition of a Limit



Formal Definition of a Limit



Formal Definition of a Limit

Remarks:

1. For $\lim_{x \to a} f(x)$ to exist, $f(x)$ must be defined on an **open** interval (α, β) containing $x = a$, except possibly at $x = a$.

2. The value of $f(a)$, if it is defined at all, does not affect the existence of the limit or its value.

3. If two functions are equal, except possibly at $x = a$, then their limiting behavior at a is the same.