The Intermediate Value Theorem

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Equator Problem

Equator Problem:

Show that at any given time there will always be two diametrically opposite points on the equator with exactly the same temperature.

Equator Problem (continued)

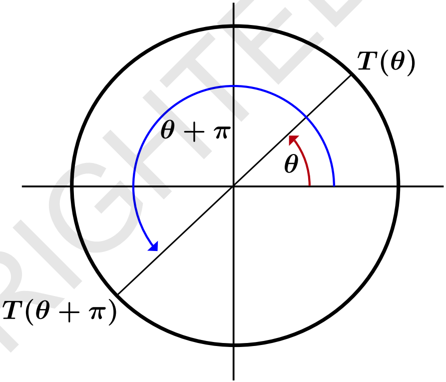
Strategy:

1) Assume the equator is a circle and that each point can be identified by an angle θ in standard position with $T(\theta)$ the temperature at the point θ.

2) Let

$$H(\theta) = T(\theta+\pi) - T(\theta).$$

3) Find a point θ_0 with $H(\theta_0) = 0$.



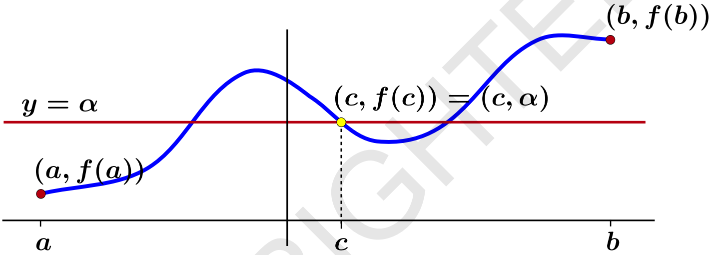
Central Problem

Central Problem: Solve

1. $f(x) = 0$
2. $f(x) = \alpha$
3. $f(x) = g(x)$

Question: Does a solution exist? If so, how do you find it?

Central Problem



Observation:

1) Assume $f(x)$ is continuous on $[a, b]$ with $f(a) < \alpha < f(b)$.
2) To get from below the line $y = \alpha$ to above the line $y = \alpha$ without creating a break, we must cross the line $y = \alpha$ at least once so there exists $c \in (a, b)$ with $f(c) = \alpha$.

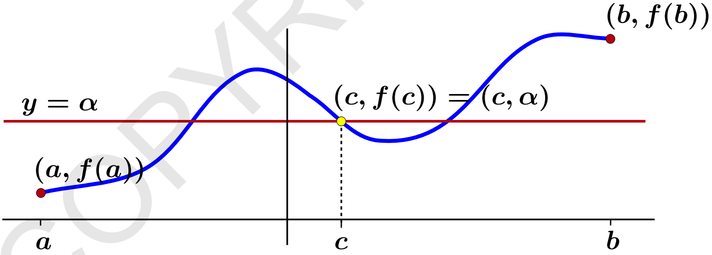
Intermediate Value Theorem (IVT)

Theorem: [The Intermediate Value Theorem (IVT)]

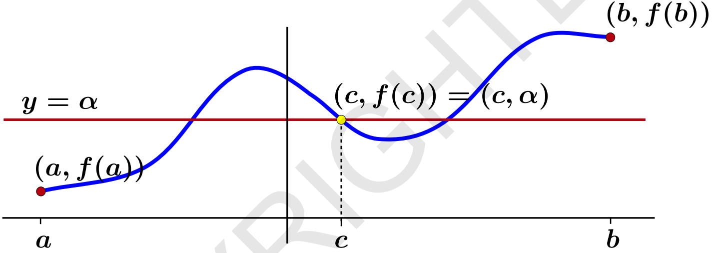
Assume that $f(x)$ is continuous on the closed interval $[a,b]$, and either

$$f(a) < \alpha < f(b) \text{ or } f(a) > \alpha > f(b).$$

Then there exists a $c \in (a,b)$ such that $f(c) = \alpha$.

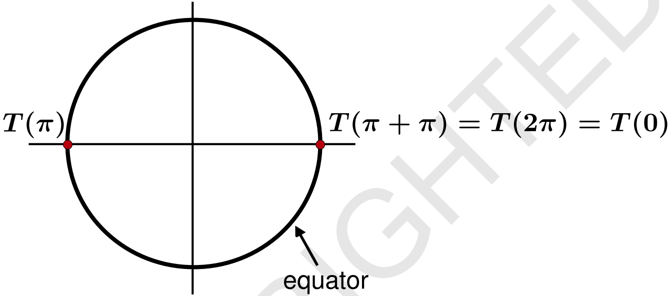


Intermediate Value Theorem (IVT)



Note: The IVT appears to be very obvious but the proof that it is true is actually quite complicated and is beyond the scope of this course.

Equator Problem



Recall:

Equator Problem:

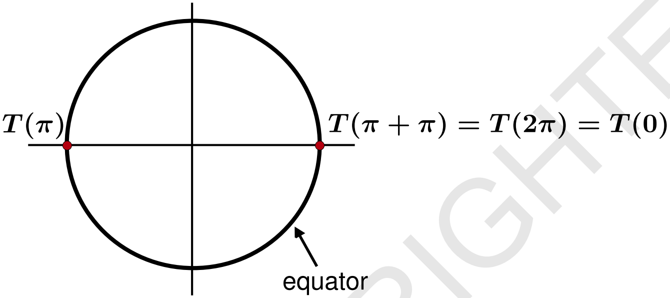
Show that at any given time there will always be two diametrically opposite points on the equator with exactly the same temperature.

Solution:

Observe $H(\theta) = T(\theta + \pi) - T(\theta)$ is continuous on $[0, \pi]$. We have

$$\begin{aligned} H(\pi) &= T(\pi + \pi) - T(\pi) \\ &= T(2\pi) - T(\pi) \\ &= T(0) - T(\pi) \\ &= -H(0). \end{aligned}$$

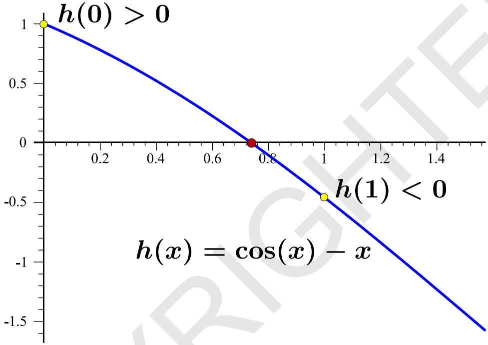
Equator Problem



Three Cases:

1) If $H(0) = 0$ we are done.
2) If $H(0) < 0$, then $H(\pi) > 0$ so the IVT gives θ_0 with $H(\theta_0) = 0$.
3) If $H(0) > 0$, then $H(\pi) < 0$ so the IVT gives θ_0 with $H(\theta_0) = 0$.

Example



Example:

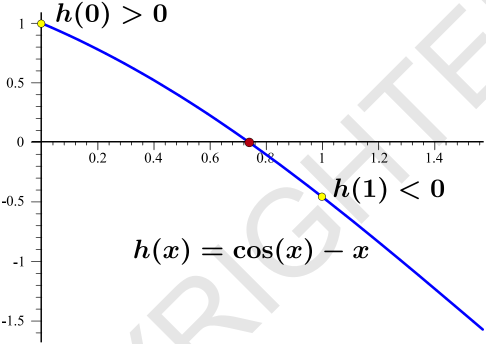
Show that there exists a $c \in (0, 1)$ such that

$$\cos(c) = c,$$

or equivalently that there exists a $c \in (0, 1)$ with $h(c) = 0$ where

$$h(x) = \cos(x) - x.$$

Example (continued)



Solution: $h(x) = \cos(x) - x$ is continuous on the closed interval $[0, 1]$ with

$$h(0) = \cos(0) - 0 = 1 > 0$$

and

$$h(1) = \cos(1) - 1 < 0.$$

By the IVT, we can conclude that there is a $0 < c < 1$ such that $h(c) = 0$ or equivalently that $\cos(c) = c$.