Horizontal Asymptotes and Limits at Infinity

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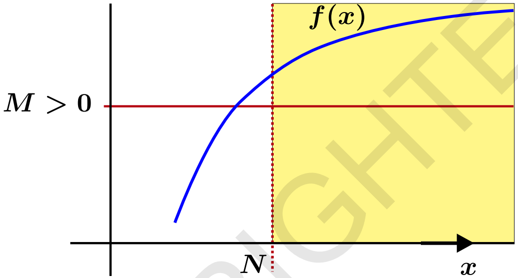
Infinite Limits at ∞



Definition: [Infinite Limits at ∞]

We say that the limit of $f(x)$ as x approaches ∞ is ∞ if for every $M > 0$ there exists a cutoff $N > 0$ such that if $x > N$, then

$$f(x) > M.$$

We write

$$\lim_{x \to \infty} f(x) = \infty.$$

Note: We can define $\lim_{x \to \infty} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = \pm\infty$ similarly.

Limits at ∞

Example: Evaluate

$$\lim_{x \to \infty} \frac{3x^3 + x}{x^2 + 1}.$$

Following the same procedure as before, we have

$$\begin{aligned} \frac{3x^3 + x}{x^2 + 1} &= \frac{x^3(3 + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} \\ &= x \cdot \left(\frac{3 + \frac{1}{x^2}}{1 + \frac{1}{x^2}} \right) \\ &\cong 3x. \end{aligned}$$

Hence

$$\lim_{x \to \infty} \frac{3x^3 + x}{x^2 + 1} = \infty.$$

Note: Formally, this means that the limit *does not exist*.

Limits of the Form $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)}$

Let

$$f(x) = \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_j x^j}{c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k}.$$

By factoring out x^j from the numerator and x^k from the denominator and rewriting the function as

$$f(x) = \frac{x^j}{x^k}\left[\frac{\frac{b_0}{x^j} + \frac{b_1}{x^{j-1}} + \frac{b_2}{x^{j-2}} + \frac{b_3}{x^{j-3}} + \cdots + b_j}{\frac{c_0}{x^k} + \frac{c_1}{x^{k-1}} + \frac{c_2}{x^{k-2}} + \cdots + c_k}\right].$$

This leads us to conclude that

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \begin{cases} \frac{b_j}{c_k} & \text{if } j = k \\ 0 & \text{if } j < k \\ \pm\infty & \text{if } j > k \end{cases}$$

where in the last case whether the limit is ∞ or $-\infty$ depends on the sign of $\frac{b_j}{c_k}$ and whether $x \to \infty$ or $-\infty$, as well as whether or not $j - k$ is even or odd.

Squeeze Theorem for Limits at $\pm\infty$

Theorem: [Squeeze Theorem for Limits at $\pm\infty$]

Assume that $g(x) \leq f(x) \leq h(x)$ for all $x \geq N$. If

$$\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$$

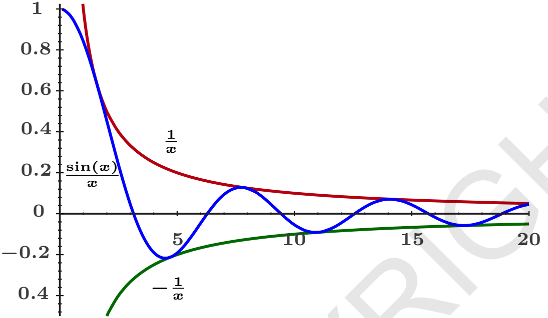
then $\lim_{x\to\infty} f(x)$ exists and it equals L.

Assume that $g(x) \leq f(x) \leq h(x)$ for all $x \leq N$. If

$$\lim_{x\to-\infty} g(x) = L = \lim_{x\to-\infty} h(x)$$

then $\lim_{x\to-\infty} f(x)$ exists and it equals L.

Limits of the form $\lim_{x \to \pm\infty} \dfrac{\sin(x)}{x}$



Example: Evaluate

$$\lim_{x \to \infty} \frac{\sin(x)}{x}.$$

Solution: Since

$$|\sin(x)| \leq 1$$

for any $x > 0$, we have

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}.$$

Now

$$\lim_{x \to \infty} -\frac{1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}$$

so the Squeeze Theorem implies that

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0.$$