Horizontal Asymptotes and Limits at Infinity

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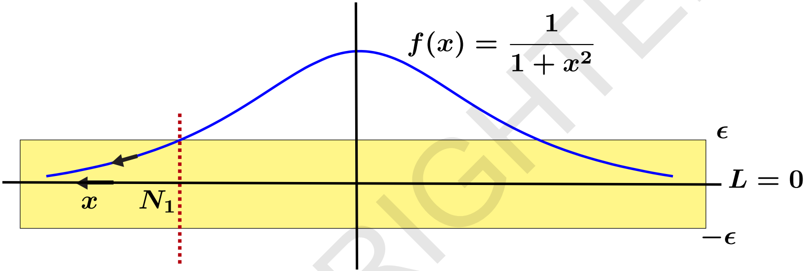
Limits at ∞

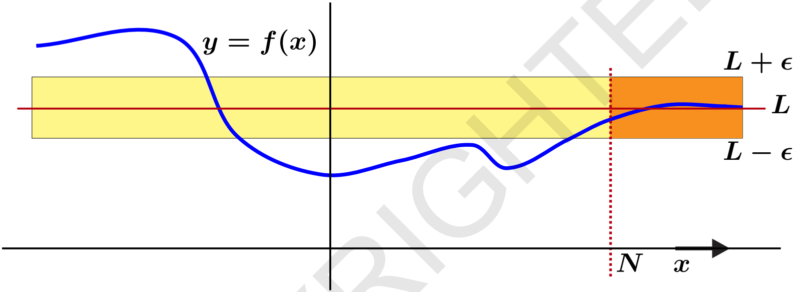


Observation: As x approaches $-\infty$, $1 + x^2$ becomes very large and hence $f(x) = \frac{1}{1+x^2}$ gets very close to 0.

More precisely, given any positive tolerance $\epsilon > 0$, we can find a cutoff N_1 so that if $x < N_1$, then $f(x)$ approximates 0 with an error less than ϵ.

We want to say that 0 is the *limit as x approaches $-\infty$* of $f(x)$.

Limits at ∞



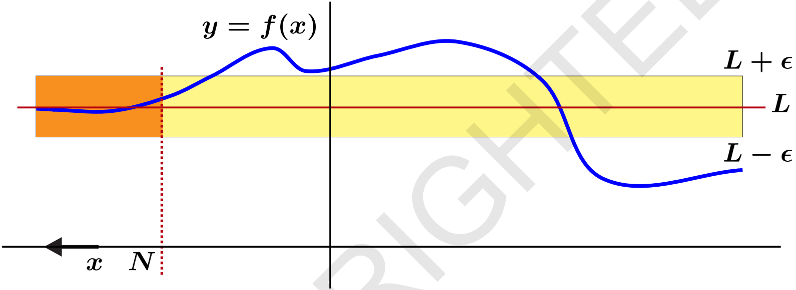
Definition: [Limit at ∞]

We say that L is the limit of $f(x)$ as x approaches ∞ if for every $\epsilon > 0$ there exists a cutoff $N > 0$ such that if $x > N$, then

$$| f(x) - L | < \epsilon.$$

We write $\lim_{x \to \infty} f(x) = L$.

Limits at ∞



Definition: [Limit at $-\infty$]

We say that L is the limit of $f(x)$ as x approaches $-\infty$ if for every $\epsilon > 0$ there exists a cutoff $N < 0$ such that if $x < N$, then

$$| f(x) - L | < \epsilon.$$

We write $\lim_{x \to -\infty} f(x) = L$.

Horizontal Asymptotes

Definition: [Horizontal Asymptote]

We say that $y = L$ is a horizontal asymptote for $f(x)$ if one of $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Note: All of the *Arithmetic Rules for Limits* hold for limits at $\pm\infty$.

Rational Functions

Example: Evaluate

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 4}{x^2 + x - 5}.$$

Observation: For polynomials and large values of x, the highest power terms dominate. Hence, for x is very large

$$\frac{2x^2 - 3x + 4}{x^2 + x - 5} \cong \frac{2x^2}{x^2} = 2.$$

We might guess that

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 4}{x^2 + x - 5} = 2.$$

Rational Functions

Example (continued): Evaluate

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 4}{x^2 + x - 5}.$$

Solution: We have for all $x > 0$,

$$\begin{aligned} \frac{2x^2 - 3x + 4}{x^2 + x - 5} &= \frac{x^2(2 - \frac{3}{x} + \frac{4}{x^2})}{x^2(1 + \frac{1}{x} - \frac{5}{x^2})} \\ &= \frac{2 - \frac{3}{x} + \frac{4}{x^2}}{1 + \frac{1}{x} - \frac{5}{x^2}} \end{aligned}$$

Then

$$\begin{aligned} \lim_{x\to\infty} \frac{2x^2 - 3x + 4}{x^2 + x - 5} &= \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{4}{x^2}}{1 + \frac{1}{x} - \frac{5}{x^2}} \\ &= \frac{2 - 0 + 0}{1 + 0 - 0} \\ &= 2. \end{aligned}$$

Rational Functions

Example: Evaluate

$$\lim_{x \to -\infty} \frac{3x^3 + x}{x^4 + 1}.$$

Solution: We have

$$\begin{aligned}
\lim_{x \to -\infty} \frac{3x^3 + x}{x^4 + 1} &= \lim_{x \to -\infty} \frac{x^3(3 + \frac{1}{x^2})}{x^4(1 + \frac{1}{x^4})} \\
&= \lim_{x \to -\infty} \left(\frac{1}{x}\right) \left(\frac{3 + \frac{1}{x^2}}{1 + \frac{1}{x^4}}\right) \\
&= 0 \cdot 3
\end{aligned}$$

Hence

$$\lim_{x \to -\infty} \frac{3x^3 + x}{x^4 + 1} = 0.$$