The Fundamental Log Limit

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Fundamental Log Limit

Important Observation:

Note that $\ln(x)$ grows much more slowly than x. For example,

$$\ln(10000) = 9.210340468.$$

This would lead us to guess that

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0.$$

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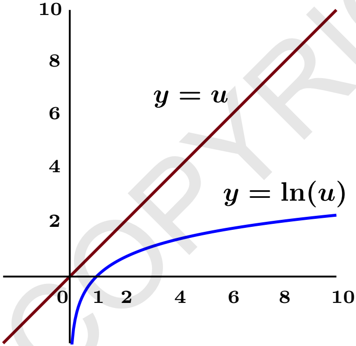
Theorem: [Fundamental Log Limit]

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0.$$



Proof:

First note that for each $u > 0$, we have

$$\ln(u) < u.$$

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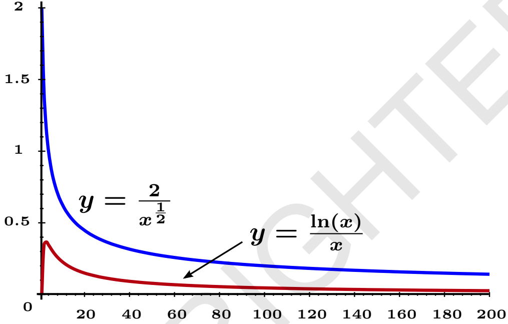

Proof (continued): If we let $u = x^{\frac{1}{2}}$ and $x \geq 1$, then

$$0 \leq \frac{\ln(x)}{x} = \frac{2\ln(x^{\frac{1}{2}})}{x} = \frac{2}{x^{\frac{1}{2}}}\left(\frac{\ln(x^{\frac{1}{2}})}{x^{\frac{1}{2}}}\right) \leq \frac{2}{x^{\frac{1}{2}}}.$$

Since

$$\lim_{x\to\infty} 0 = 0 = \lim_{x\to\infty} \frac{2}{x^{\frac{1}{2}}},$$

the Squeeze Theorem shows that

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0.$$

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Question: The Fundamental Log Limit shows that $\ln(x)$ grows more slowly than x. What are the relative growth rates of $\ln(x)$ versus that of $\sqrt{x}$ or $x^{\frac{1}{100}}$?

Note: We have

$$\ln(10000) = 9.210340468$$

and

$$10000^{\frac{1}{100}} = 1.096478196.$$

Does $x^{\frac{1}{100}}$ grow more slowly than $\ln(x)$? In fact, we claim

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{\frac{1}{100}}} = 0.$$

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Example: If $p > 0$, then

$$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = 0.$$

To show this, write

$$\frac{\ln(x)}{x^p} = \frac{\frac{1}{p}\ln(x^p)}{x^p}.$$

Since p is a constant, it follows that

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)}{x^p} &= \lim_{x\to\infty} \frac{\frac{1}{p}\ln(x^p)}{x^p} \\ &= \frac{1}{p}\cdot\left(\lim_{x\to\infty}\frac{\ln(x^p)}{x^p}\right) \\ &= \frac{1}{p}\cdot 0 \\ &= 0\end{aligned}$$

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Example: Evaluate

$$\lim_{x\to\infty} \frac{\ln(x^p)}{x}.$$

Solution: This is a simple variant of the Fundamental Log Limit since

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x^p)}{x} &= \lim_{x\to\infty} \frac{p\ln(x)}{x} \\ &= p \cdot \left(\lim_{x\to\infty} \frac{\ln(x)}{x}\right) \\ &= 0\end{aligned}$$

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Example: Evaluate

$$\lim_{x\to\infty} \frac{\ln(x^{40})}{x^{\frac{1}{1000}}}.$$

Solution: We know that

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln(x^{40})}{x^{\frac{1}{1000}}} &= \lim_{x\to\infty} \frac{40\ln(x)}{x^{\frac{1}{1000}}} \\ &= 40 \cdot \left(\lim_{x\to\infty} \frac{\ln(x)}{x^{\frac{1}{1000}}} \right) \\ &= 0 \end{aligned}$$

Note: If $f(x) = \frac{\ln(x^{40})}{x^{\frac{1}{1000}}}$, then

$$f(1000000) = 545.0381916.$$

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Example: Let $p > 0$. Evaluate

$$\lim_{x\to\infty} \frac{x^p}{e^x}.$$

Solution: Let $u = e^x$. Then $x = \ln(u)$ and

$$\frac{x^p}{e^x} = \frac{(\ln(u))^p}{u} = \left(\frac{\ln(u)}{u^{\frac{1}{p}}}\right)^p.$$

Note that if $x \to \infty$, then $u = e^x \to \infty$ and we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^p}{e^x} &= \lim_{u\to\infty} \left(\frac{\ln(u)}{u^{\frac{1}{p}}}\right)^p \\
&= \left(\lim_{u\to\infty} \frac{\ln(u)}{u^{\frac{1}{p}}}\right)^p \\
&= 0^p \\
&= 0.
\end{aligned}$$

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Notation: Assume that $f(x)$ and $g(x)$ are positive functions. We write

$$f(x) \ll g(x) \quad \text{as } x \to \infty$$

if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Summary:

$$\ln(x) \ll x^p \ll e^x$$

as $x \to \infty$.