The Fundamental Trig Limit

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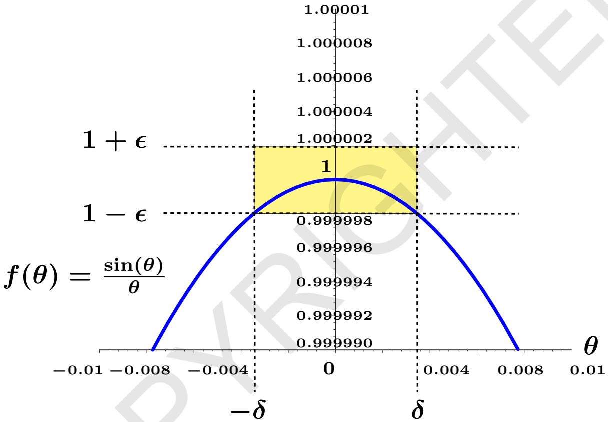
Fundamental Trig Limit



Theorem: [Fundamental Trig Limit] (FTL)

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

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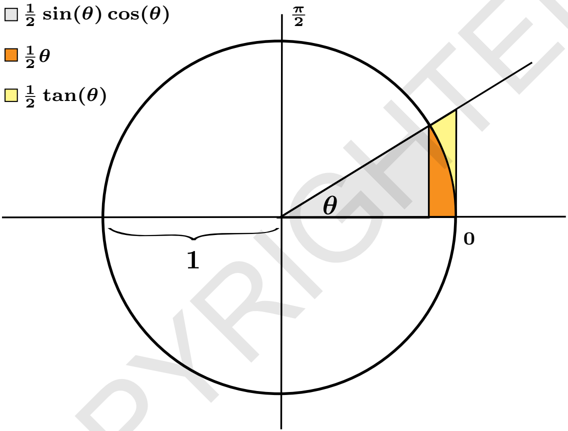

Observation: Let $0 < \theta < \frac{\pi}{2} \Rightarrow$

$$\frac{\sin(\theta)\cos(\theta)}{2} < \frac{\theta}{2} < \frac{\sin(\theta)}{2\cos(\theta)}.$$

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Proof of the FTL: Let $0 < \theta < \frac{\pi}{2}$. Then

$$\Rightarrow \quad \frac{\sin(\theta)\cos(\theta)}{2} < \frac{\theta}{2} < \frac{\sin(\theta)}{2\cos(\theta)}$$

$$\Rightarrow \quad \sin(\theta)\cos(\theta) < \theta < \frac{\sin(\theta)}{\cos(\theta)}$$

$$\Rightarrow \quad \cos(\theta) < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}$$

$$\Rightarrow \quad \cos(\theta) < \frac{\sin(\theta)}{\theta} < \frac{1}{\cos(\theta)}$$

Now

$$\lim_{\theta\to 0^+} \cos(\theta) = 1 = \lim_{\theta\to 0^+} \frac{1}{\cos(\theta)} \Rightarrow \lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

That $\lim_{\theta\to 0^-} \frac{\sin(\theta)}{\theta} = 1$ follows since $\frac{\sin(\theta)}{\theta}$ is an even function.

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Example 1: Show that $\lim_{\theta \to 0} \frac{\tan(\theta)}{\theta} = 1$.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\tan(\theta)}{\theta} &= \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta \cos(\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos(\theta)} \\
&= 1 \cdot 1 \\
&= 1.
\end{aligned}$$

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Note: The FTL says that if "$\diamond$" is "small", then

$$\sin(\diamond) \cong \diamond \cong \tan(\diamond).$$

Example 2: Find $\lim_{\theta \to 0} \frac{\sin(3\theta)}{\sin(\theta)}$.

Intuitive Solution:

If θ is small, so is 3θ. Therefore

$$\frac{\sin(3\theta)}{\sin(\theta)} \cong \frac{3\theta}{\theta} = 3.$$

Hence

$$\lim_{\theta \to 0} \frac{\sin(3\theta)}{\sin(\theta)} = 3?$$

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Example 2 (revisited): Find $\lim_{\theta \to 0} \frac{\sin(3\theta)}{\sin(\theta)}$.

Solution: Note that

$$\lim_{\theta \to 0} \frac{\sin(3\theta)}{3\theta} = 1 = \frac{1}{\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}} = \lim_{\theta \to 0} \frac{\theta}{\sin(\theta)}.$$

Since

$$\frac{\sin(3\theta)}{\sin(\theta)} = 3\left(\frac{\sin(3\theta)}{3\theta}\right)\left(\frac{\theta}{\sin(\theta)}\right),$$

we get

$$\begin{aligned} \lim_{\theta \to 0} \frac{\sin(3\theta)}{\sin(\theta)} &= \lim_{\theta \to 0} 3\left(\frac{\sin(3\theta)}{3\theta}\right)\left(\frac{\theta}{\sin(\theta)}\right) \\ &= 3\left(\lim_{\theta \to 0} \frac{\sin(3\theta)}{3\theta}\right)\left(\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)}\right) \\ &= 3(1)(1) \\ &= 3. \end{aligned}$$

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Example 3: Find $\lim_{\theta \to 0} \dfrac{\theta \sin(\pi\theta)}{\tan^2(2\theta)}$.

Intuitive Solution:

If θ is small so are 2θ and $\pi\theta$. Therefore

$$\frac{\theta \sin(\pi\theta)}{\tan^2(2\theta)} \cong \frac{\theta \cdot \pi\theta}{(2\theta)^2} = \frac{\pi}{4}.$$

Hence

$$\lim_{\theta \to 0} \frac{\theta \sin(\pi\theta)}{\tan^2(2\theta)} = \frac{\pi}{4}?$$

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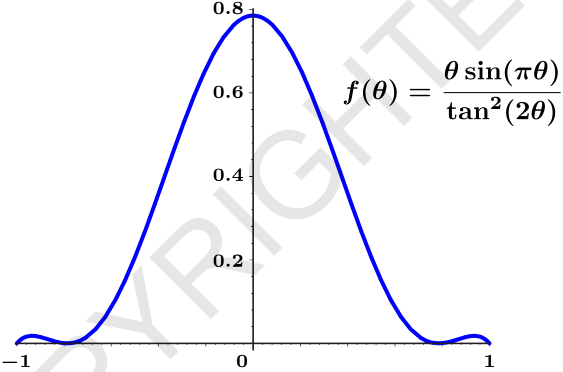

Note:

$$\frac{\pi}{4} \cong .7853981635.$$

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Example 4: Show that $\lim_{\theta \to 0} \frac{\theta \sin(\pi\theta)}{\tan^2(2\theta)} = \frac{\pi}{4}$.

Solution:

$$\lim_{\theta \to 0} \frac{\theta \sin(\pi\theta)}{\tan^2(2\theta)} =$$

$$\begin{aligned}
&= \left[\lim_{\theta \to 0} \frac{2\theta}{\tan(2\theta)}\right] \cdot \left[\lim_{\theta \to 0} \frac{\sin(\pi\theta)}{\pi\theta}\right] \cdot \left[\lim_{\theta \to 0} \frac{2\theta}{\tan(2\theta)}\right] \cdot \left[\frac{\pi}{2 \cdot 2}\right] \\
&= 1 \cdot 1 \cdot 1 \cdot \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$