The Extreme Value Theorem

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Global Extrema

Goal: Many important applications of mathematics require you to find the largest or smallest value of a function over a given set.

Definition: [Global Maxima and Global Minima]

Suppose that $f : I \to \mathbb{R}$, where I is a non-degenerate interval.

- We say that c is a *global maximum for* $f(x)$ *on* I if $c \in I$ and $f(x) \leq f(c)$, for all $x \in I$.
- We say that c is a *global minimum for* $f(x)$ *on* I if $c \in I$ and $f(x) \geq f(c)$, for all $x \in I$.
- We say that c is an *global extremum for* $f(x)$ *on* I if it is either a global maximum or a global minimum for $f(x)$ on I.

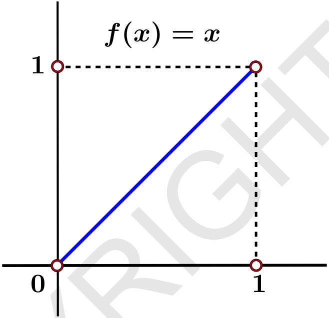
Global Extrema

Question:

Given a function $f(x)$ defined on a non-degenerate interval I, do there exist points $c_1, c_2 \in I$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in I$?

That is, does $f(x)$ achieve both a global maximum and global minimum on I?

Example

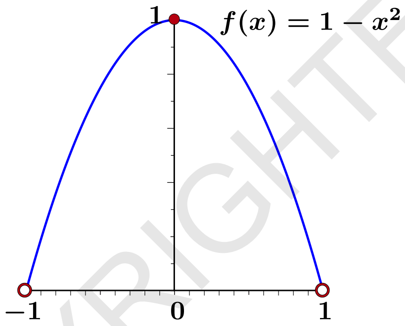


Example:

Let $f(x) = x$. Since the open interval $(0, 1)$ has no largest or smallest value, $f(x)$ has no global maximum or global minimum on $(0, 1)$.

Key Observation: This function seems to want to have a maximum and a minimum at the end points $x = 0$ and $x = 1$ of the open interval $(0, 1)$.

Example



Example:

Let $f(x) = 1 - x^2$ on the open interval $(-1, 1)$. Then $f(x)$ has no global minimum on $(-1, 1)$, but $f(x)$ does have a global maximum on the interval $(-1, 1)$ at $x = 0$.

Observation: Again, this function does seem to want to achieve it's minimum at the missing end points of the open interval $(-1, 1)$.

Extreme Value Theorem

Theorem: [The Extreme Value Theorem (EVT)]

Suppose that $f(x)$ is continuous on $[a, b]$. There exists c_1 and $c_2 \in [a, b]$ such that

$$f(c_1) \leq f(x) \leq f(c_2)$$

for all $x \in [a, b]$.

Question: Is continuity important?

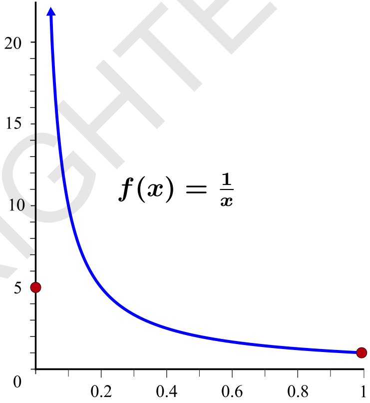
Example

Example:

Let

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 5 & \text{if } x = 0 \end{cases}$$

Then $f(x)$ has a global minimum on $[0, 1]$ at $x = 1$, but it has no global maximum on $[0, 1]$.



Note: This example does not contradict the EVT because $f(x)$ is **not** continuous on $[0, 1]$.

Extreme Value Theorem

Observation: The EVT ensures that a continuous function $f(x)$ defined on a closed interval $[a, b]$ achieves its global maximum and minimum on $[a, b]$, but it does not tell us how to find these values.

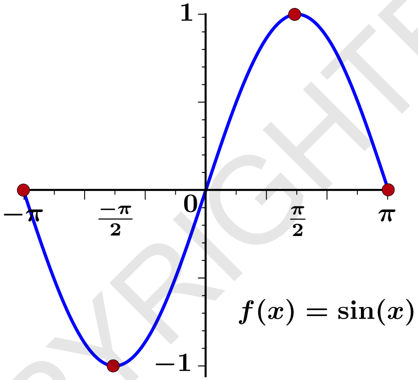
Important Fact: Assume that $f(x)$ has either a global maximum or global minimum at $c \in [a, b]$. Then either

1) c is an end point of the interval $[a, b] \Rightarrow c = a$ or $c = b$

or

2) c is **not** an end point of the interval $[a, b] \Rightarrow c \in (a, b)$.

Example



Example:

The function $f(x) = \sin(x)$ assumes its maximum and minimum values on $[-\pi, \pi]$ at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$, respectively.

Extreme Value Theorem

Important Fact:

Continuous functions can behave very differently on closed intervals than they do on open intervals.