Types of Discontinuities

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Continuity

Definition: [Continuity at a Point]

We say that $f(x)$ is continuous at $x = a$ if

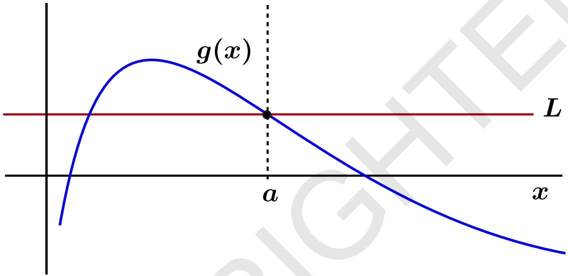
1) $\lim_{x \to a} f(x)$ exists, and
2) $\lim_{x \to a} f(x) = f(a)$.

Notation: $D(f) = \{x_0 \mid f(x)$ is not continuous at $x_0\}$.

Observation: We have $x_0 \in D(f)$ if

1. $\lim_{x \to a} f(x)$ exists but is not $f(a)$, or
2. $\lim_{x \to a} f(x)$ does not exist.

Removable Discontinuity



Definition: [Removable Discontinuity]

We say that $f(x)$ has a *removable discontinuity* at $x = a$ if $\lim\limits_{x \to a} f(x)$ exists but is not $f(a)$.

Define

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a, \\ \lim\limits_{x \to a} f(x) & \text{if } x = a. \end{cases}$$

Then $g(x)$ **removes the discontinuity** of $f(x)$ at $x = a$.

Essential Discontinuity

Definition: [Essential Discontinuity]

We say that $f(x)$ has an *essential discontinuity* at $x = a$ if $\lim\limits_{x \to a} f(x)$ does not exist.

Note: The discontinuity is called essential because there is no way to eliminate it by redefining the value of $f(x)$ at $x = a$.

There are three basic types of essential discontinuities:

1. Jump discontinuities.
2. Vertical asymptotes.
3. Oscillatory discontinuities.

Jump Discontinuity

Definition: [Jump Discontinuity]

We say that $f(x)$ has a *jump discontinuity* at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

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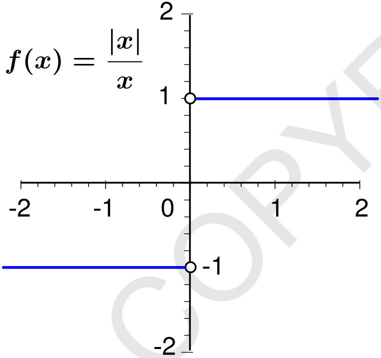

Examples:

1. $f(x) = \frac{|x|}{x}$ at $x = 0$.

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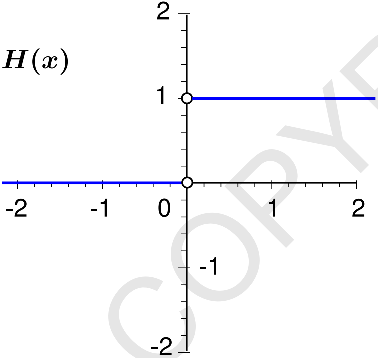


Examples:

1. $f(x) = \frac{|x|}{x}$ at $x = 0$.

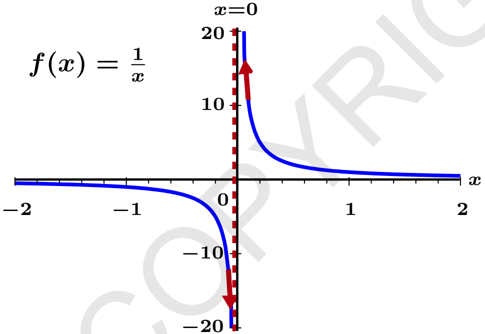
2. Heaviside function:

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Vertical Asymptote

Definition: [Vertical Asymptote]

We say that $f(x)$ has a *vertical asymptote* at $x = a$ if at least one of $\lim\limits_{x \to a^{\pm}} f(x) = \pm\infty$ holds.

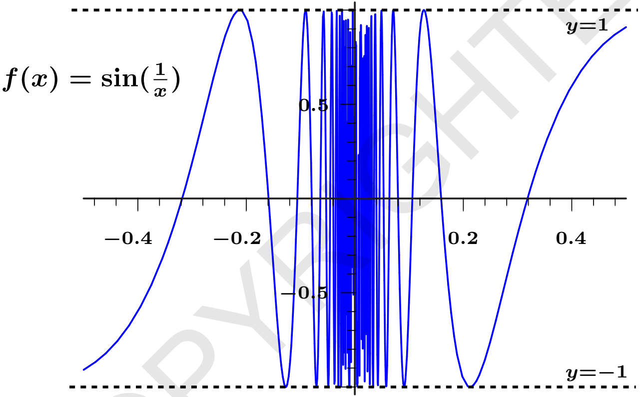


Example:

$f(x) = \frac{1}{x}$ at $x = 0$.

Note: This is sometimes called an *infinite jump discontinuity*.

Oscillatory Discontinuities



Example: $f(x) = \sin(\frac{1}{x})$ has an oscillatory discontinuity at $x = 0$.