Curve Sketching
Part I

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Curve Sketching

Goal:

To use information about limits and continuity to sketch the graphs of various functions.

Strategy: Basic Curve Sketching

Step 1: Determine the domain of $f(x)$.

Step 2: Determine any symmetries that the graph may have. In particular, test to see if the function is either even or odd.

Step 3: Determine, if possible, where the function changes sign and plot these points.

Step 4: Find any discontinuity points for $f(x)$.

Step 5: Evaluate the relevant one-sided and two-sided limits at the points of discontinuity and identify the nature of the discontinuities. In particular, indicate any removable discontinuity with a small circle to denote the hole.

Step 6: Using the information from step (5), draw any vertical asymptotes.

Step 7: Find any horizontal asymptotes by evaluating the limits of the function at $\pm\infty$, if applicable. Draw the horizontal asymptotes on your plot.

Step 8: Finally, use the information you have gathered above to construct as accurate a sketch as possible for the graph of the given function. It is often helpful to plot a few sample points as a guide.

Example: Basic Curve Sketching

Example: Sketch the graph of

$$\frac{xe^x}{x^3 - x}.$$

Solution:

Step 1: This function is defined everywhere except when the denominator is zero:

$$x^3 - x = x(x-1)(x+1) = 0.$$

That is, everywhere except when $x = 0$ and $x = \pm 1$.

Step 2: Since

$$f(-x) = \frac{-xe^{-x}}{-x^3 + x} \neq f(x) \text{ and } f(-x) = \frac{xe^{-x}}{x^3 - x} \neq -f(x)$$

$f(x)$ is neither even nor odd. In fact, there are no obvious symmetries.

Example: Basic Curve Sketching

Solution (continued):

Step 3: We first observe that

$$f(x) = \frac{xe^x}{x^3 - x} = \frac{e^x}{x^2 - 1}$$

for all $x \neq 0$. Since e^x is never 0, the function is never 0.

The IVT tells us that we could only have a sign change at a point of discontinuity.

Example: Basic Curve Sketching

Solution (continued):

Step 4:

$$f(x) = \frac{xe^x}{x^3 - x}$$

The function is the ratio of two continuous functions. Therefore, $f(x)$ is discontinuous only at $x = 0$, $x = -1$ and $x = 1$ since these are the only points where the denominator is 0.

Example: Basic Curve Sketching

Solution (continued):

Step 5:

1) e^x is always positive and since $f(x) = \frac{xe^x}{x^3-x} = \frac{e^x}{x^2-1}$ for all $x \neq 0$, there is no sign change at $x = 0$.

2) $f(x)$ goes from positive to negative as we move across $x = -1$ and then from negative to positive as we cross $x = 1$ (moving left to right).

Example: Basic Curve Sketching

Solution (continued):

Step 5 (continued):

3) Since for $x \neq 0$, we have

$$f(x) = \frac{xe^x}{x^3 - x} = \frac{e^x}{x^2 - 1}$$

this gives us that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x}{x^2 - 1} = -1.$$

So $x = 0$ is a removable discontinuity.

Example: Basic Curve Sketching

Solution (continued):

Step 5 (continued):

4) $x = 1$ and $x = -1$ are both vertical asymptotes for $f(x)$.
5) Since $f(x) > 0$ if $x > 1$ or $x < -1$, and $f(x) < 0$ if $x \neq 0$ and $-1 < x < 1$, we get that

$$\lim_{x \to 1^+} f(x) = \infty,$$

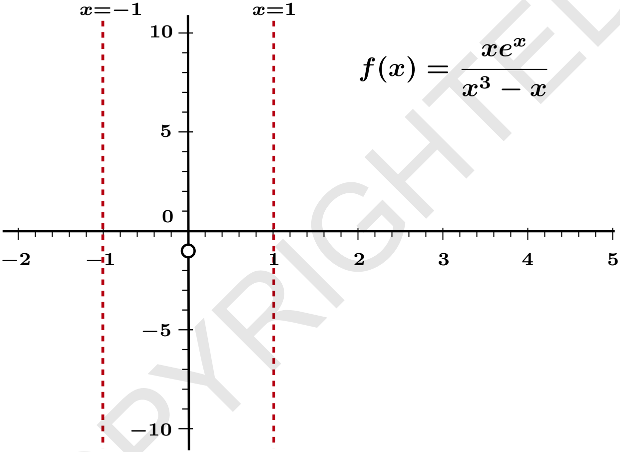
$$\lim_{x \to 1^-} f(x) = -\infty,$$

and

$$\lim_{x \to -1^+} f(x) = -\infty,$$

$$\lim_{x \to -1^-} f(x) = \infty.$$

Example: Basic Curve Sketching



Solution (continued):

Step 6: Draw the vertical asymptotes and indicate the removable discontinuities on the plot.

Example: Basic Curve Sketching

Solution (continued):

Step 7:

$$f(x) = \frac{xe^x}{x^3 - x}$$

1) Since e^x grows much more rapidly than any polynomial for large positive values of x, we have

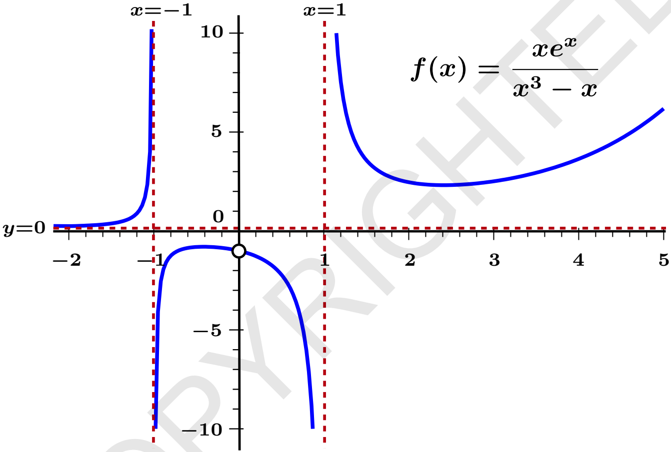
$$\lim_{x \to \infty} f(x) = \infty.$$

2) But since e^x becomes very small for large negative values of x, we have

$$\lim_{x \to -\infty} f(x) = 0.$$

Thus, $y = 0$ is a horizontal asymptote as $x \to -\infty$.

Example: Basic Curve Sketching



Solution (continued):

Step 8: Sketch the graph using all of this information.