The Bisection Algorithm for Approximating Zeros

Created by

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Central Problem

Central Problem: Solve $f(x) = 0$.

Question: Does a solution exist? If so, how do you find it?

The Intermediate Value Theorem

Theorem: [Intermediate Value Theorem (IVT)]

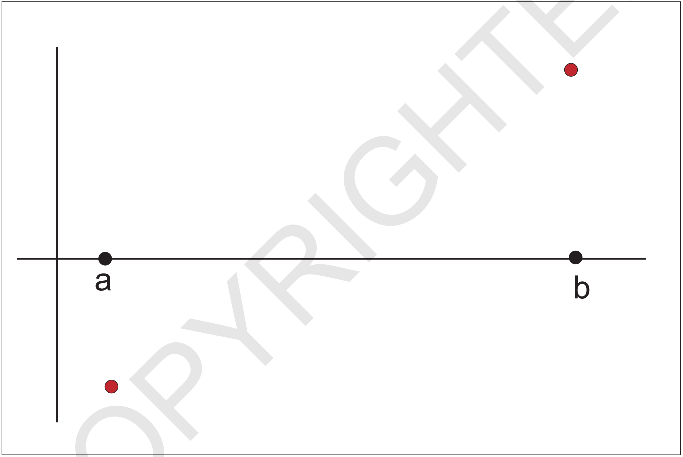
Assume that $f(x)$ is continuous on $[a, b]$ and that $f(a) \cdot f(b) < 0$. Then there exists $c \in (a, b)$ such that

$$f(c) = 0.$$

Note: We have $f(a) \cdot f(b) < 0$ if and only if $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$.

Question: How can we estimate c within a given error?

The Strategy



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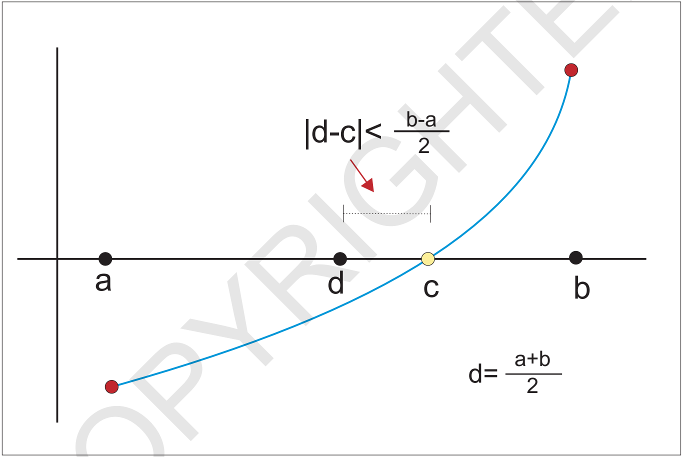
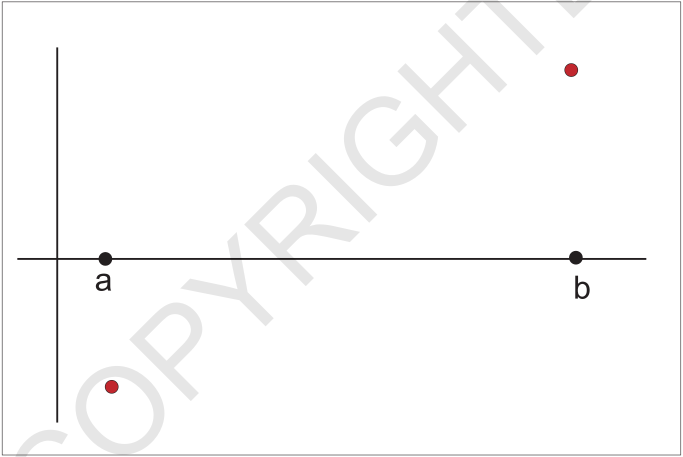


Use $d = \frac{a+b}{2}$ as an estimate. $\Rightarrow$ **Error** $= \mid d - c \mid < \frac{b-a}{2}$.

The Strategy

Can we do better?



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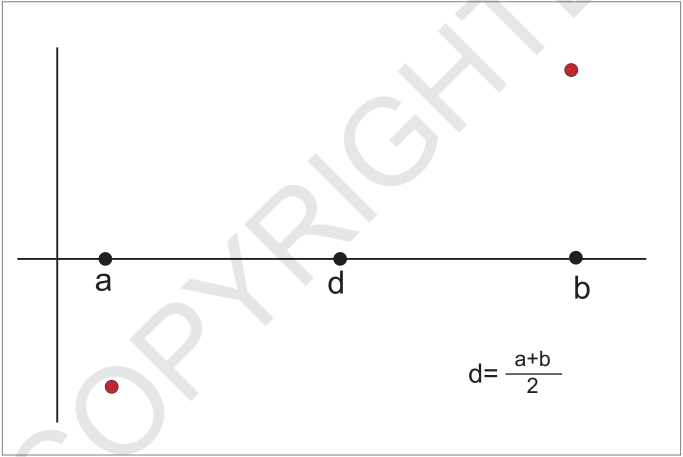

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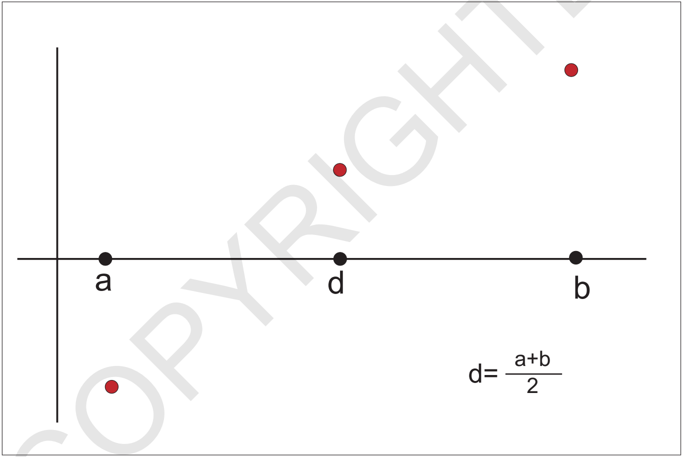

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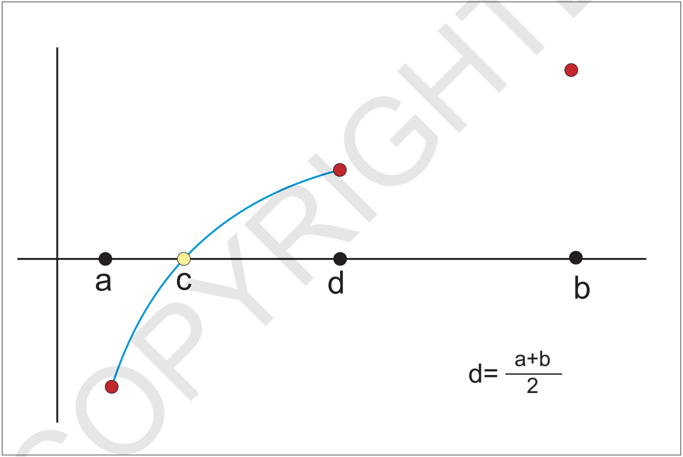

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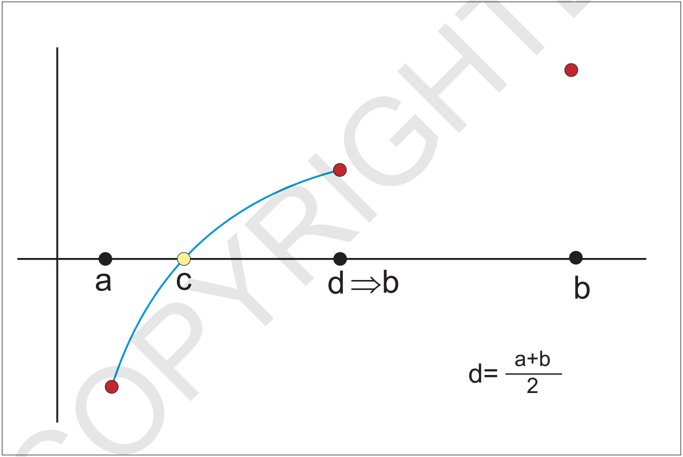

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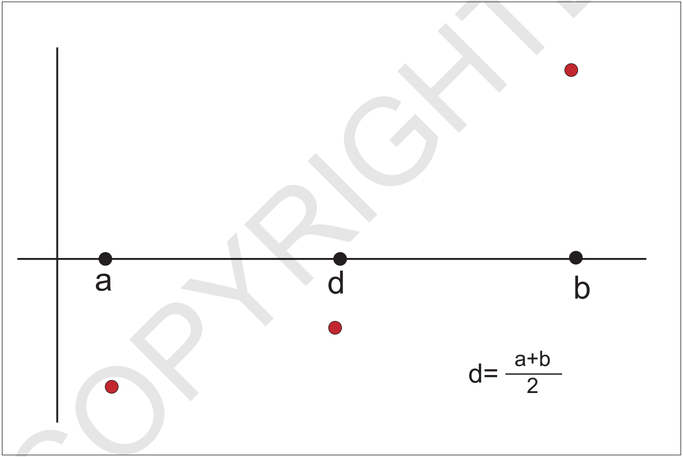

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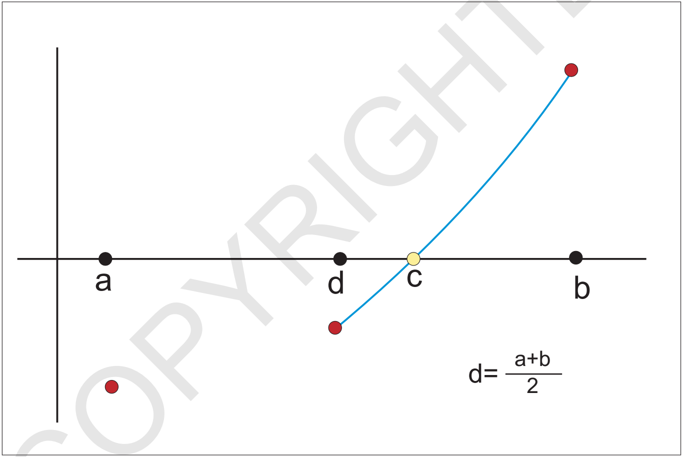

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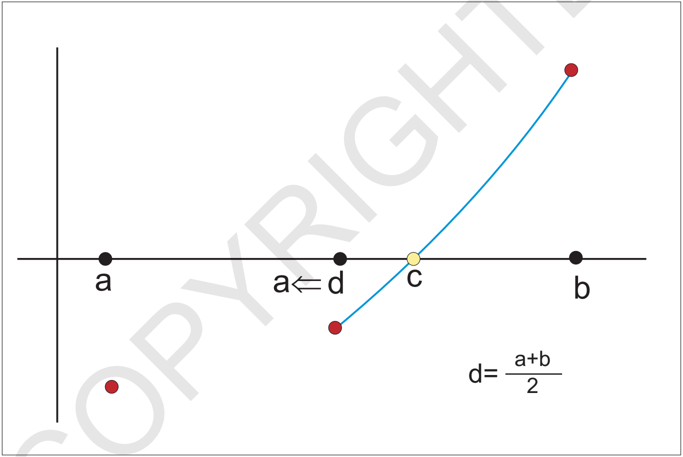

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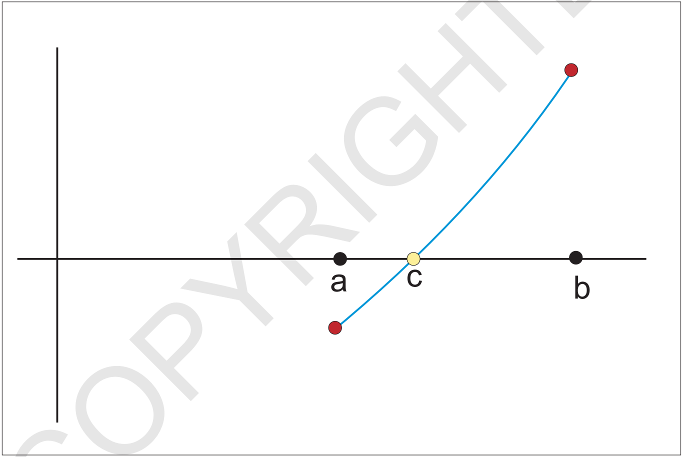

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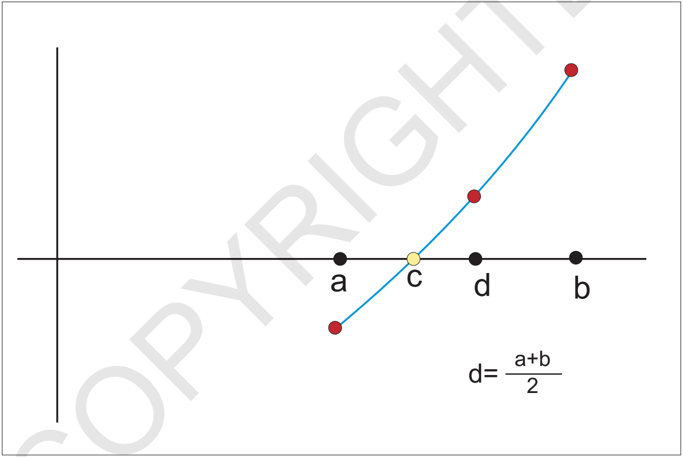

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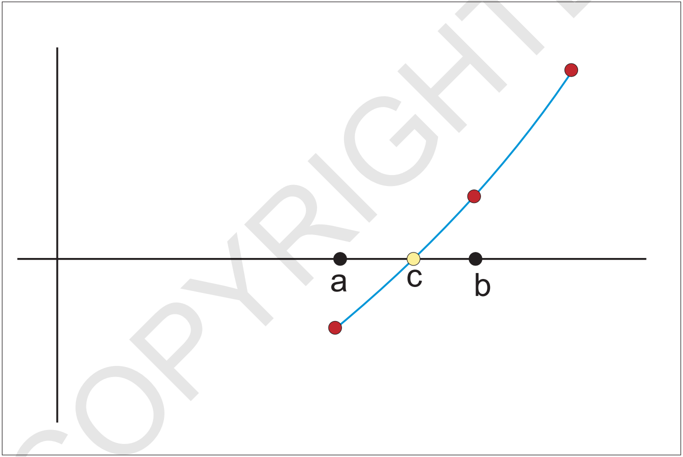

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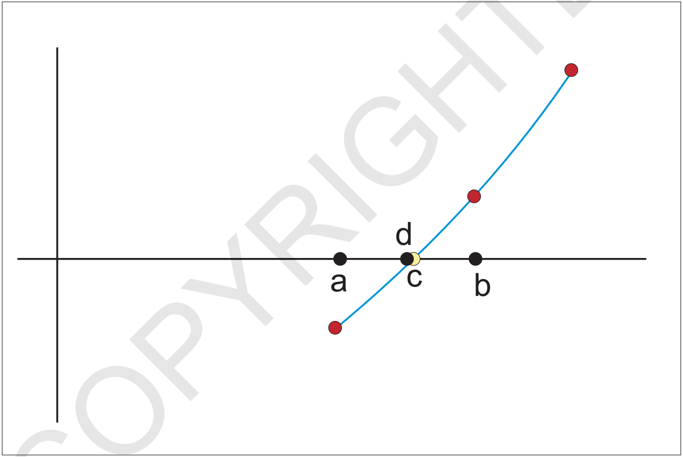

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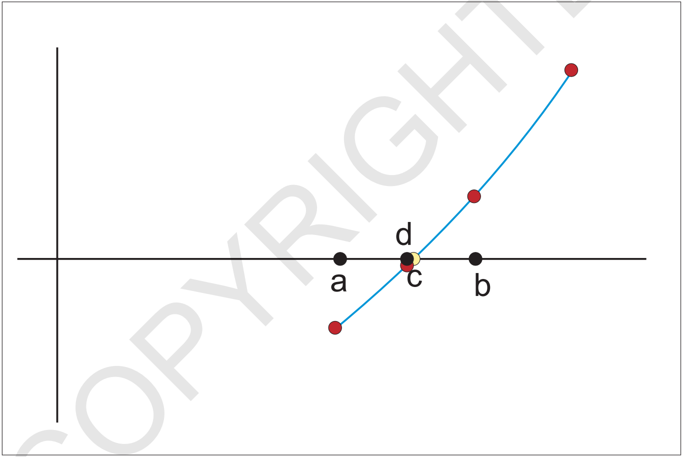

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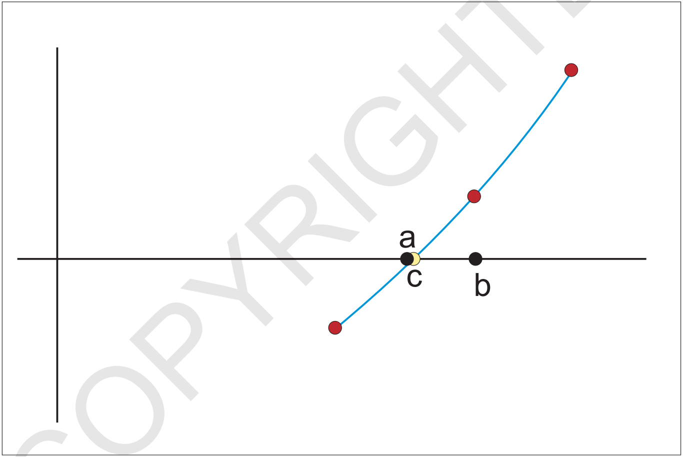


Bisection Algorithm for Approximating Zeros

Problem: Given a continuous function $f(x)$ and an $\epsilon > 0$, find an approximation d to a solution $x = c$ for the equation

$$f(x) = 0$$

such that

$$|d - c| < \epsilon.$$

Bisection Algorithm for Approximating Zeros:

Step 1: Find a, b so that $f(a) \cdot f(b) < 0$.

Step 2: Let $d = \frac{a+b}{2}$.

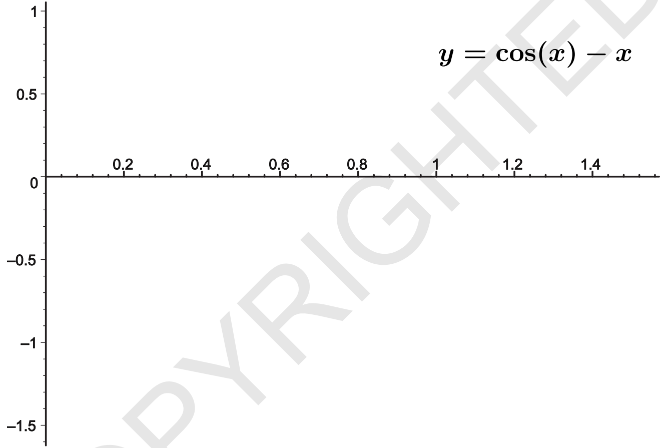
Step 3: If $f(d) = 0$ then stop, else go to Step 4.

Step 4: If $\frac{b-a}{2} < \epsilon$ then stop, else go to Step 5.

Step 5: If $f(a) \cdot f(d) < 0$ then $b = d$ and go to Step 2, else go to Step 6.

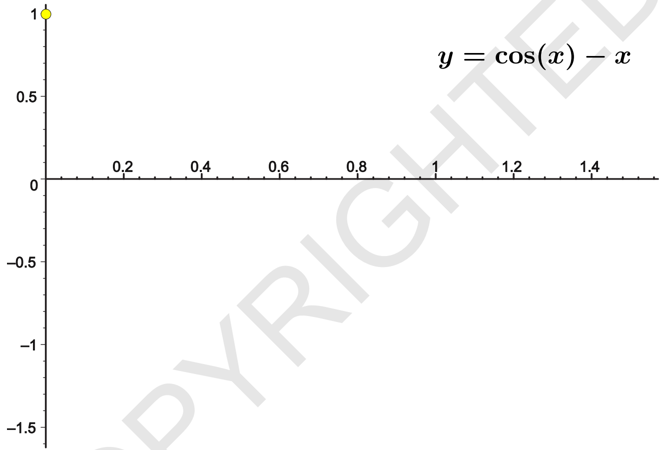
Step 6: $a = d$ and go to Step 2.

Example



Problem: Estimate c so that $\cos(c) = c \Leftrightarrow \cos(c) - c = 0$ with an error less than $\frac{1}{2^{25}}$.

Example



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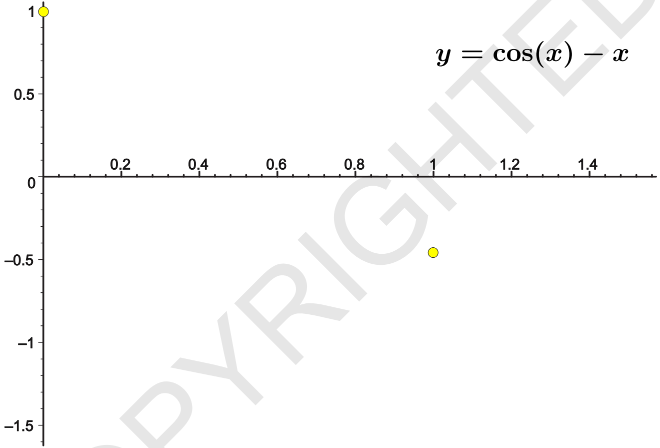

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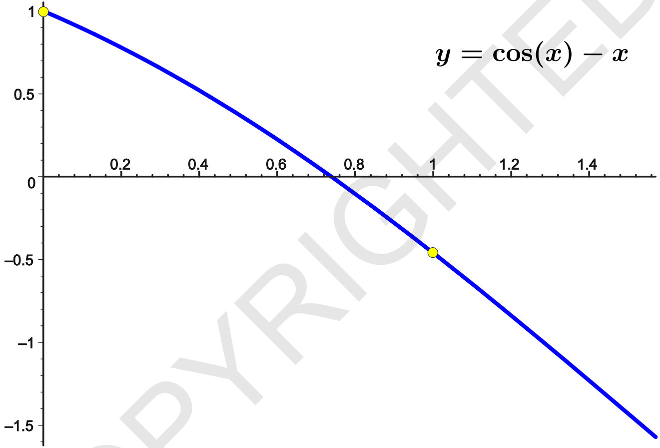

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Start with $a = 0$ and $b = 1$

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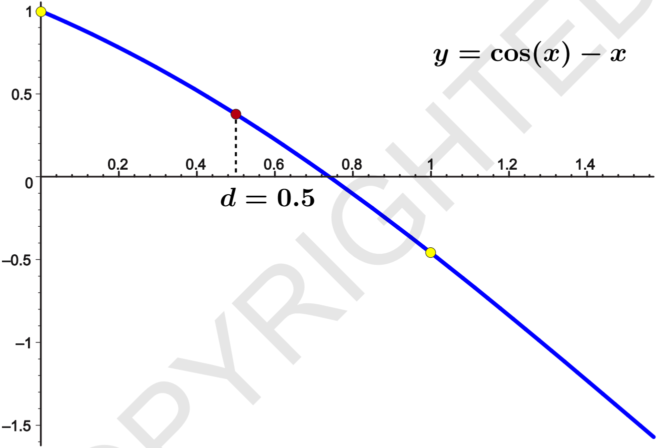
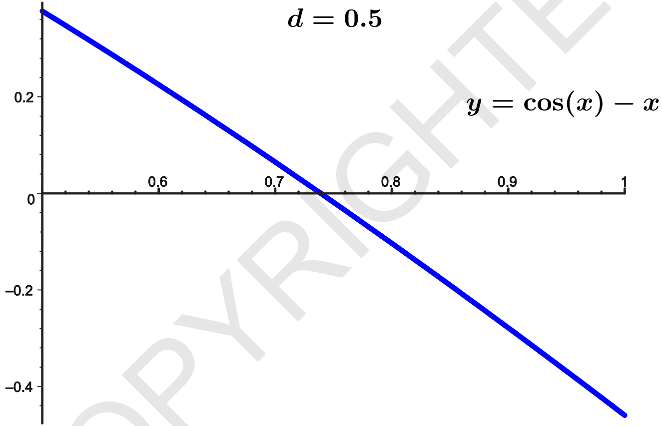


Problem: Estimate c so that $\cos(c) = c \Leftrightarrow \cos(c) - c = 0$ with an error less than $\frac{1}{2^{25}}$.

Start with $a = 0$ and $b = 1 \Rightarrow d = 0.5$.

Example



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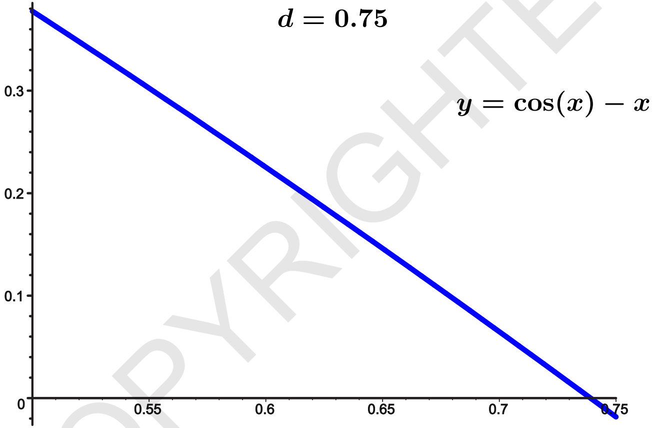

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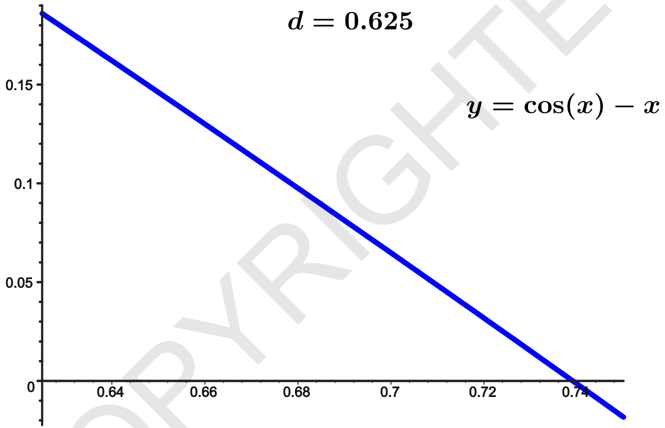

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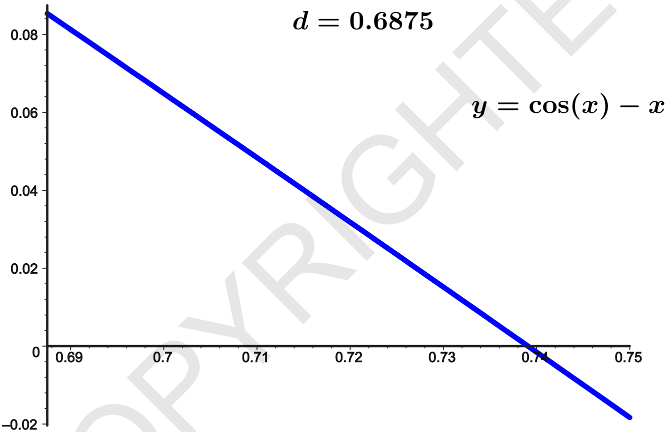

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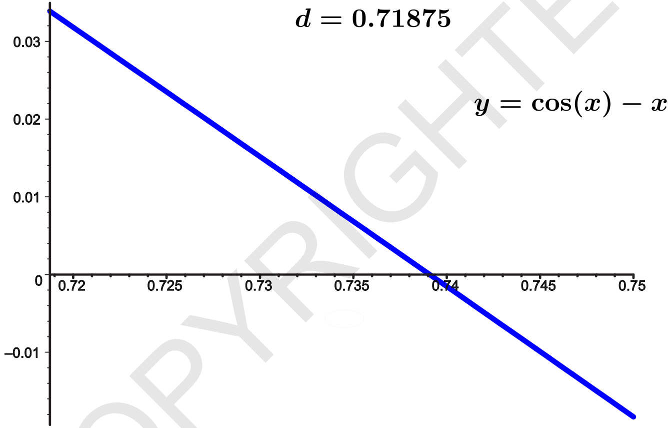
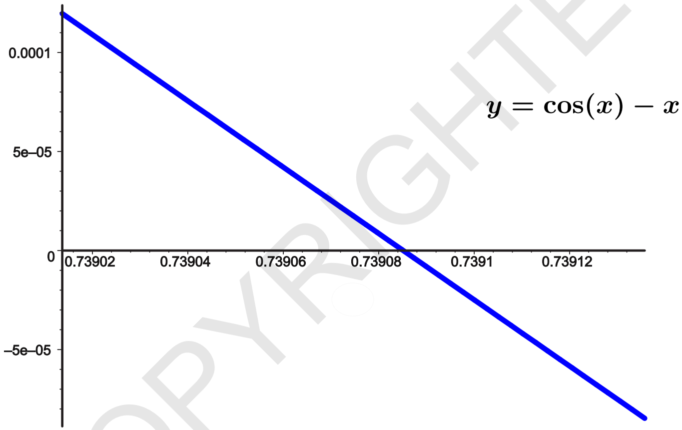


$d = 0.71875$
$y = \cos(x) - x$
0.03
0.02
0.01
0
–0.01
0.72
0.725
0.73
0.735
0.74
0.745
0.75

Example



Solution: After 25 steps we get $c \cong d = \mathbf{0.7390851230}$ with an error that is less than $\frac{1}{2^{25}}$.