Arithmetic Rules for Limits of Functions

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Arithmetic of Limits

Recall:

Theorem: [Arithmetic of Limits for Sequences]

Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

1) If $c \in \mathbb{R}$, then $\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n = cL$.
2) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$.
3) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = LM$.
4) If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $M \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n} = \frac{1}{M}$.
5) If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $M \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$.
6) If $\lim_{n\to\infty} \frac{a_n}{b_n} = S$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Arithmetic of Limits

Theorem: [Arithmetic of Limits for Functions]

Assume that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

1) If $c \in \mathbb{R}$, then $\lim_{x \to a} cf(x) = cL$.
2) $\lim_{x \to a} (f(x) + g(x)) = L + M$.
3) $\lim_{x \to a} f(x)g(x) = LM$.
4) If $M \neq 0$, then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$.
5) If $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.
6) If $\lim_{x \to a} \frac{f(x)}{g(x)} = S$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$.

Arithmetic of Limits

Theorem

If

1) $\lim_{x \to a} g(x) = 0$ and
2) $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ exists,

then $\lim_{x \to a} f(x)$ exists and equals 0.

Proof:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \frac{f(x)}{g(x)} \cdot g(x) \\ &= \left(\lim_{x \to a} \frac{f(x)}{g(x)} \right) \cdot \left(\lim_{x \to a} g(x) \right) \\ &= L \cdot 0 \\ &= 0. \end{aligned}$$

Limits of Polynomial Functions

Theorem: [Limits of Polynomial Functions]

Let $p(x)$ be any *polynomial*. That is,

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

Then

$$\lim_{x \to a} p(x) = p(a).$$

Rational Functions

Definition: [Rational Function]

A *rational function* is a function of the form

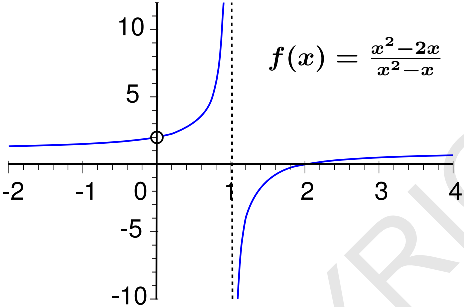
$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

Problem: How do we find

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{P(x)}{Q(x)}?$$

Example



Example: Find $\lim_{x\to 1} \frac{x^2-2x}{x^2-x}$.

Solution: Note that

$$\lim_{x\to 1} x^2 - x = 0,$$

but

$$\lim_{x\to 1} x^2 - 2x = -1 \neq 0,$$

which implies that $\lim_{x\to 1} \frac{x^2-2x}{x^2-x}$ does not exist.

Observation: The graph of $f(x)$ shows that the function is “unbounded” near $x = 1$.

Limits of Rational Functions

The following algorithm can be used to compute

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{P(x)}{Q(x)}.$$

Step 1) If $Q(a) \neq 0$, then $\lim_{x \to a} f(x) = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)} = f(a)$.

Step 2) If $Q(a) = 0$ and $P(a) \neq 0$, then the limit does not exist.

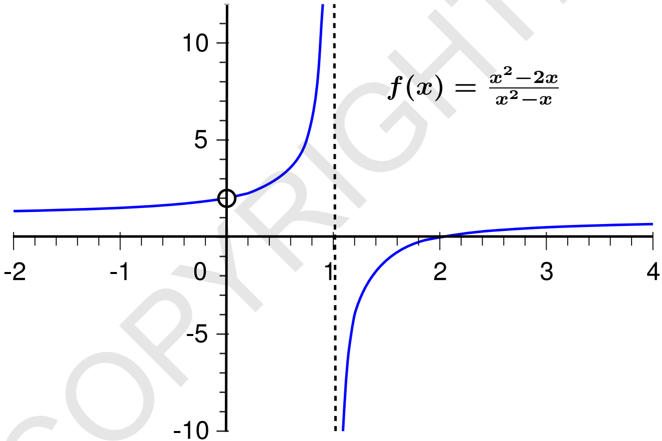
Step 3) If $Q(a) = 0 = P(a)$, then both $P(x)$ and $Q(x)$ can be factored into $Q(x) = (x-a)Q_1(x)$ and $P(x) = (x-a)P_1(x)$

$\Rightarrow f(x) = \frac{(x-a)P_1(x)}{(x-a)Q_1(x)} = \frac{P_1(x)}{Q_1(x)} = f_1(x)$ for any $x \neq a$.

Repeat Step 1) with $f_1(x) = \frac{P_1(x)}{Q_1(x)}$.

Examples

Recall: The graph of $f(x) = \frac{x^2-2x}{x^2-x} = \frac{P(x)}{Q(x)}$ appears as follows:



Examples

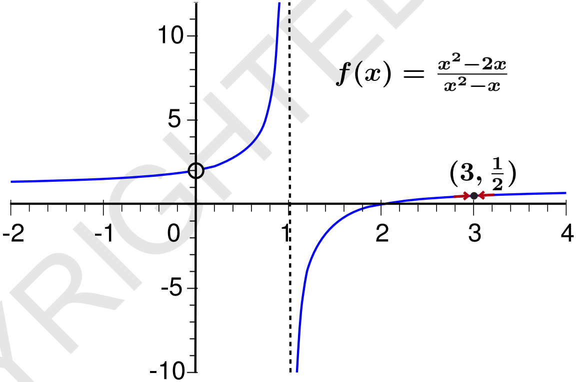
Example: Find $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 2x}{x^2 - x}$.

Solution: Note that

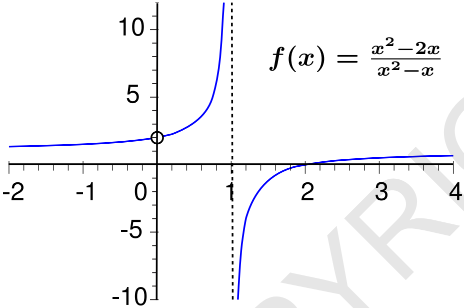
$$\begin{aligned} Q(3) &= (3)^2 - 3 \\ &= 6 \neq 0. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{x \to 3} f(x) &= f(3) \\ &= \frac{3^2 - 2(3)}{3^2 - 3} \\ &= \frac{1}{2}. \end{aligned}$$



Example



$f(x) = \frac{x^2-2x}{x^2-x}$

Example: Find $\lim_{x \to 1} \frac{x^2-2x}{x^2-x}$.

Solution: Since

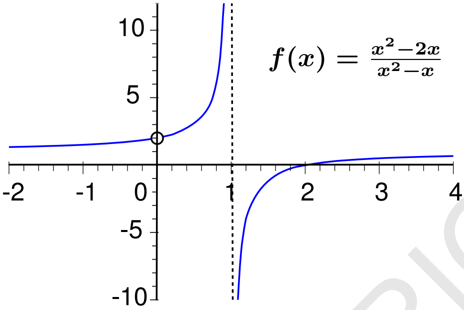
$$Q(1) = 1^2 - 1 = 0,$$

but

$$P(1) = 1^2 - 2(1) = -1 \neq 0,$$

it follows that $\lim_{x \to 1} \frac{x^2-2x}{x^2-x}$ does not exist.

Examples



Example: Find $\lim_{x\to 0} \frac{x^2-2x}{x^2-x}$.

Solution: Note that

$$P(0) = 0 = Q(0).$$

So both $P(x) = x^2 - x$ and $Q(x) = x^2 - 2x$ contain the factor $(x - 0) = x$.

Hence

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{(x)(x-2)}{(x)(x-1)} \\ &= \lim_{x\to 0} \frac{x-2}{x-1} \\ &= \frac{\lim_{x\to 0}(x-2)}{\lim_{x\to 0}(x-1)} \\ &= \frac{-2}{-1} = 2.\end{aligned}$$