Arithmetic Rules for Continuity

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Sequential Characterization of Continuity

Recall:

Theorem: [Sequential Characterization of Continuity]

The following are equivalent:

1) $f(x)$ is continuous at $x = a$.
2) If $\{x_n\}$ is a sequence with $x_n \to a$, then $\lim_{n\to\infty} f(x_n) = f(a)$.

Arithmetic Rules for Continuity

Theorem: [Arithmetic Rules for Continuity]

If f and g are both continuous at $x = a$, then we have the following:

1. $(cf)(x) = c \cdot f(x)$ is continuous at $x = a$ for all $c \in \mathbb{R}$.
2. $(f + g)(x) = f(x) + g(x)$ is continuous at $x = a$.
3. $(fg)(x) = f(x)g(x)$ is continuous at $x = a$.
4. If $g(a) \neq 0$, then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$.

Polynomials and Rational Functions

Remark:

1) Let $P(x)$ be any polynomial. Then $P(x)$ is continuous at any $a \in \mathbb{R}$.

2) Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function. Then $f(x)$ is continuous at $x = a$ if and only if $Q(a) \neq 0$.

3) If a rational function $f(x) = \frac{P(x)}{Q(x)}$ is not continuous at $x = a$, then either $x = a$ is a removable discontinuity or $x = a$ is a vertical asymptote.

Continuity of Composite Functions

Theorem: [Continuity of Composite Functions]

If $f(x)$ is continuous at $x = a$ and $g(y)$ is continuous at $y = f(a)$, then $h(x) = g \circ f(x)$ is continuous at $x = a$.

Proof: Let $x_n \to a$. Then $f(x_n) \to f(a)$ since $f(x)$ is continuous at $x = a$. Since $g(y)$ is continuous at $f(a)$,

$$h(x_n) = g \circ f(x_n) = g(f(x_n)) \to g(f(a)) = g \circ f(a) = h(a).$$

Therefore, $h(x)$ is continuous at $x = a$ by the *Sequential Characterization of Continuity*.

Example: The function

$$f(x) = e^{x^2 \sin(x)}$$

is continuous on $\mathbb{R}$.