Approximating Roots of a Polynomial

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Roots of a Polynomial

Definition: [Roots of a Polynomial]

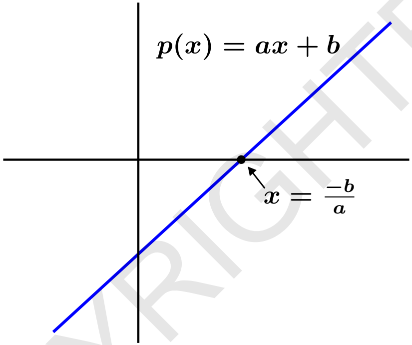
If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial, then a root of $p(x)$ is any number c such that $p(c) = 0$.

Problem: How do you know if a polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

has any real roots and if it does, how do you find them?

Roots of a Linear Polynomial



Example: **Linear Polynomials**

If $p(x) = ax + b$, where $a \neq 0$, the only real root is given by

$$x = \frac{-b}{a}.$$

Roots of a Quadratic Polynomial

Example: Quadratic Polynomials

If $p(x) = ax^2 + bx + c$, the quadratic formula tells us that $p(x) = 0$ if and only if

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, $p(x)$ has a real root if and only if $b^2 - 4ac \geq 0$.

Roots of a 5th Degree Polynomial

Example: Does $p(x) = x^5 + x - 1$ have any real roots and if so, how can we find one?

Remarks:

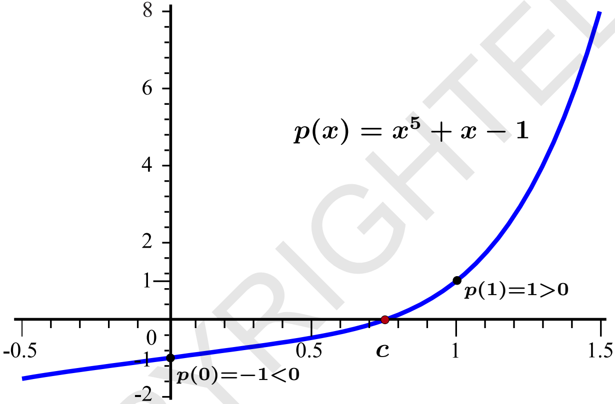
1) There is no formula to find the roots of a generic fifth degree polynomial.
2) We have

   $$p(0) = 0^5 + 0 - 1 = -1 < 0 \quad \text{and} \quad p(1) = 1^5 + 1 - 1 = 1 > 0.$$

3) Since $p(x)$ is continuous on the closed interval $[0, 1]$, the IVT guarantees that there is a point c with $0 < c < 1$ such that $p(c) = 0$.
4) Since $p'(x) = 5x^4 + 1 > 0$, $p(x)$ has only one root.

Question: Can the IVT help us find c?

Roots of a 5th Degree Polynomial

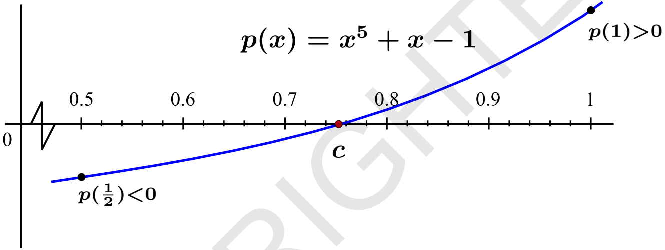


Step 1: We know that if $p(x) = x^5 + x - 1$, then

$$p(0) < 0 \text{ and } p(1) > 0$$

so $c \in (0, 1)$.

Roots of a 5th Degree Polynomial

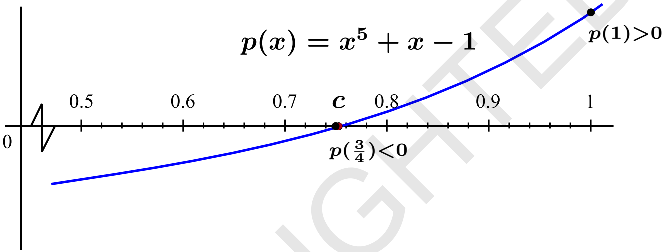


Step 2: Test the midpoint $d_1 = \frac{1}{2}$ between 0 and 1 to get

$$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^5 + \frac{1}{2} - 1 = -\frac{15}{32} < 0.$$

We have $p\left(\frac{1}{2}\right) < 0$ and $p(1) > 0$, so $c \in \left(\frac{1}{2}, 1\right)$.

Roots of a 5th Degree Polynomial



Step 3: Test the midpoint $d_2 = \frac{3}{4} = .75$ between $\frac{1}{2}$ and 1.

If $p(.75) > 0$, then since $p(.5) < 0$, the root would be in the interval $[.5, .75]$.

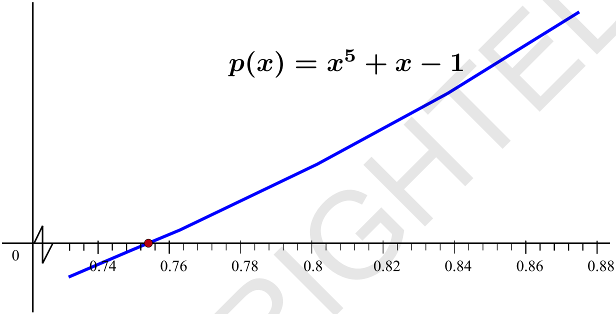
If $p(.75) < 0$, the root would be in the interval $[.75, 1]$ since we know that $p(1) > 0$.

In fact,

$$p(.75) = -.0126953 < 0$$

so the root is in the interval $[.75, 1]$.

Roots of a 5th Degree Polynomial



Step 4: Find the new midpoint:

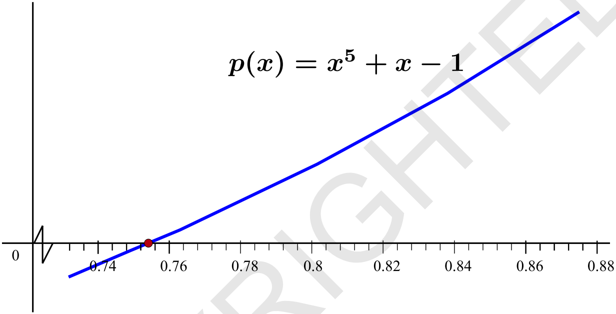
$$d_3 = \frac{1 + .75}{2} = .875$$

Test this new midpoint to find that

$$p(.875) = .3879089 > 0$$

Since the sign change now occurs between 0.75 and 0.875, the next interval of interest is $[.75, .875]$.

Roots of a 5th Degree Polynomial



Step 5: Again, find the new midpoint:

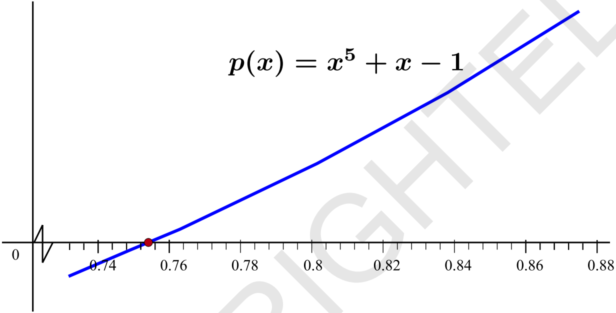
$$d_4 = \frac{.75 + .875}{2} = .8125$$

We get that

$$p(.8125) = .1665926 > 0$$

so we know that the root lies in the interval $[.75, .8125]$.

Roots of a 5th Degree Polynomial



Step 6: Continue by finding the next midpoint

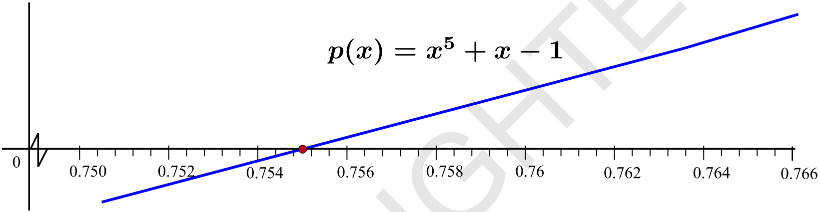
$$d_5 = \frac{.75 + .8125}{2} = .78125$$

and then determine that

$$p(.78125) = .0722883 > 0.$$

Since $p(.78125)$ is the same sign as $p(.8125)$, we replace 0.8125 with 0.78125 to give us the new interval $[.75, .78125]$.

Roots of a 5th Degree Polynomial



Step 7: The next midpoint becomes

$$d_6 = \frac{.75 + .78125}{2} = .765625$$

We have

$$p(.765625) = .0287006 > 0$$

so the root is in the interval $[.75, .765625]$.

Roots of a 5th Degree Polynomial

Step 8: The next midpoint is

$$d_7 = \frac{.75 + .765625}{2} = .7578125$$

Evaluating $p(x)$ at this point gives

$$p(.7578125) = .007737 > 0$$

so the sign change occurs between $x = .75$ and $x = .7578125$

Step 9: One more iteration of the procedure gives us a new midpoint

$$d_8 = \frac{.75 + .7578125}{2} = .75390625$$

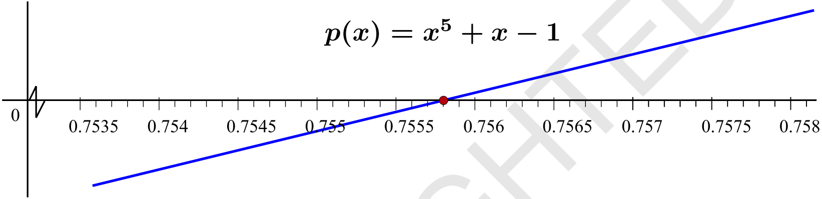
with

$$p(.75390625) = -.002544 < 0.$$

Replace 0.75 as the new left-hand endpoint with 0.75390625 so that

$$0.75390625 < c < 0.7578125$$

Roots of a 5th Degree Polynomial



Observation: We have not found c, but we know that

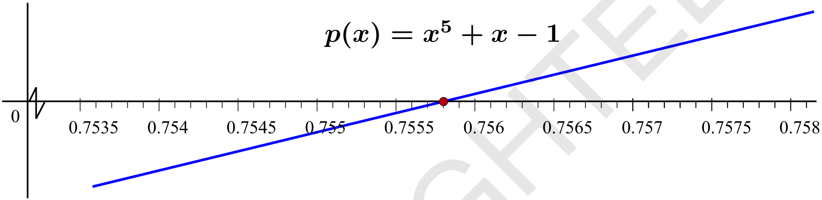
$$0.75390625 < c < 0.7578125$$

The length of this interval is

$$.7578125 - 0.75390625 = .00390625 = \frac{1}{256} = \frac{1}{2^8}.$$

Note: The original interval had length 1 and we have run through 8 iterations of the procedure with each iteration producing a new interval exactly $\frac{1}{2}$ of the length of the previous interval.

Roots of a 5th Degree Polynomial



Conclusion: If we want to make the final estimate d_9 of the root, the midpoint of the previous two endpoints is

$$c \cong d_9 = \frac{0.75390625 + 0.7578125}{2} = 0.755859375$$

The error in the estimate is at most the maximum distance from the final estimate to each of the two endpoints in the final interval. That is,

$$|d_9 - c| = |\ 0.755859375 - c\ | \leq \frac{1}{2^9} = \frac{1}{512}.$$