Limits of Sequences

Created by

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Definition of a Limit of a Sequence

Recall:

Formal Definition: [Limit of a Sequence]

We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$| a_n - L | < \epsilon.$$

In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

We may also say $\{a_n\}$ converges to L and write $a_n \to L$.

If no such L exists, we say that $\{a_n\}$ *diverges*.

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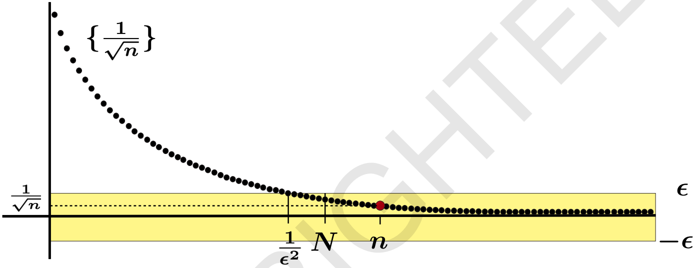
Example 1



Example 1: Show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Let $\epsilon > 0$. We need to find a cutoff N that satisfies the definition of the limit.

- If $\frac{1}{\epsilon^2} < n \Rightarrow \frac{1}{n} < \epsilon^2 \Rightarrow \frac{1}{\sqrt{n}} < \epsilon$.
- Hence, if $\frac{1}{\epsilon^2} < N$, then $n \geq N \Rightarrow \left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon$.

Therefore, we have shown the limit is 0.

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Example 2

Example 2: It can be shown that

$$\lim_{n\to\infty} \frac{3n+2}{4n+3} = \frac{3}{4}.$$

Find a cutoff N so that if $n \geq N$, then

$$\left|\frac{3n+2}{4n+3} - \frac{3}{4}\right| < \frac{1}{1000}.$$

Solution: Observe that

$$\begin{aligned}\left|\frac{3n+2}{4n+3} - \frac{3}{4}\right| &= \left|\frac{12n+8}{16n+12} - \frac{12n+9}{16n+12}\right| \\ &= \left|\frac{-1}{16n+12}\right| \\ &= \frac{1}{16n+12}\end{aligned}$$

We want

$$\frac{1}{16n+12} < \frac{1}{1000} \Rightarrow 1000 < 16n+12 \Rightarrow 61.75 < n, \text{ so } N = 62.$$

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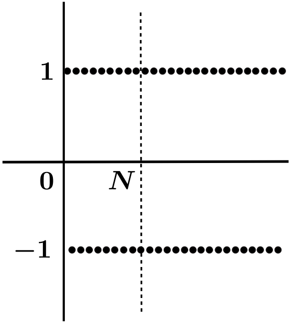
Example 3



Example 3:

Consider $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$.

Does $\{(-1)^{n+1}\}$ have a limit?

Is $\lim_{n\to\infty} \{(-1)^{n+1}\} = 1$?

Is $\lim_{n\to\infty} \{(-1)^{n+1}\} = -1$?

Or both?

Solution: Assume $\lim_{n\to\infty} \{(-1)^{n+1}\} = L$ and $\epsilon = \frac{1}{2}$. Choose the cutoff N such that if $n \geq N$ then,

$$| (-1)^{n+1} - L | < \frac{1}{2}.$$

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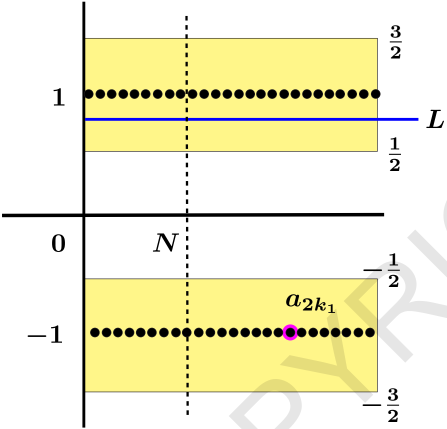
Example 3 (continued)



Pick $k_1 \in \mathbb{N}$ such that $2k_1 \geq N$.

Then

$$a_{2k_1} = -1$$

$$\implies |-1 - L| < \frac{1}{2}$$

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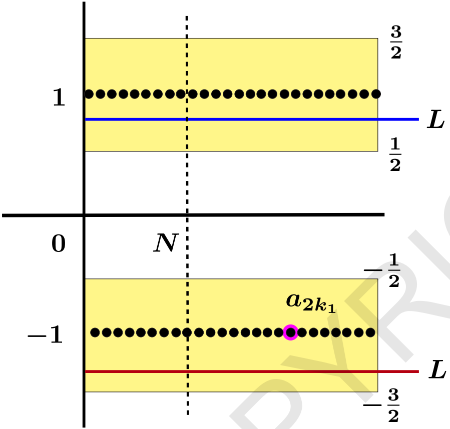
Example 3 (continued)



Pick $k_1 \in \mathbb{N}$ such that $2k_1 \geq N$.

Then

$$\begin{aligned} a_{2k_1} &= -1 \\ &\Longrightarrow \quad |-1 - L| < \frac{1}{2} \\ &\Longrightarrow \quad L \in (-\frac{3}{2}, -\frac{1}{2}) \end{aligned}$$

Hence, $L \in (-\frac{3}{2}, -\frac{1}{2})$ and $L \in (\frac{1}{2}, \frac{3}{2})$ which is impossible.

Therefore, $\{(-1)^{n+1}\}$ has no limit!

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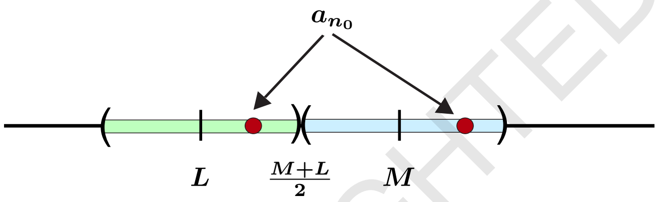
Uniqueness of Limits



Problem: Can $\{a_n\}$ have two different limits?

Assume $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ with $L < M$.

Consider $\frac{M+L}{2}$. Let $\epsilon = \frac{M-L}{2}$.

Consider a_{n_0}. If n_0 is large enough, then

$$a_{n_0} \in (M - \epsilon, M + \epsilon)$$

and

$$a_{n_0} \in (L - \epsilon, L + \epsilon)$$

which is impossible!

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Uniqueness of Limits

Theorem: [Uniqueness of Limits]

Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Then

$$L = M.$$

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Example 4

Note: It is often difficult to tell if a sequence converges and if so, what its limit might be.

Example 4: Consider the recursively defined sequence

$$a_1 = 1, \qquad a_{n+1} = \cos(a_n).$$

Does $\{a_n\}$ converge? If so, what is $\lim_{n\to\infty} a_n$?

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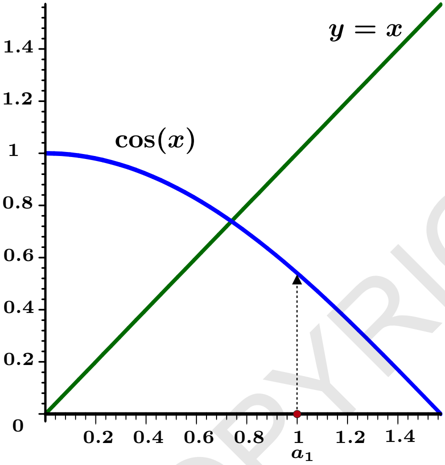
Example 4 (continued)



$a_1 = 1,$

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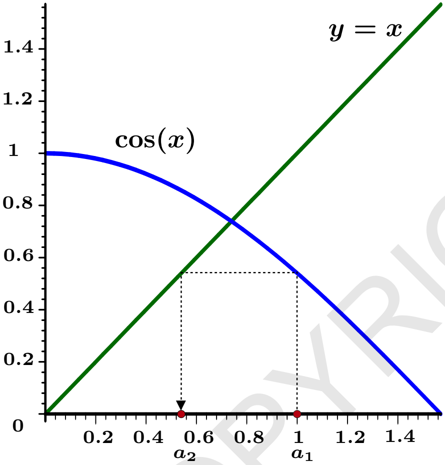
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$

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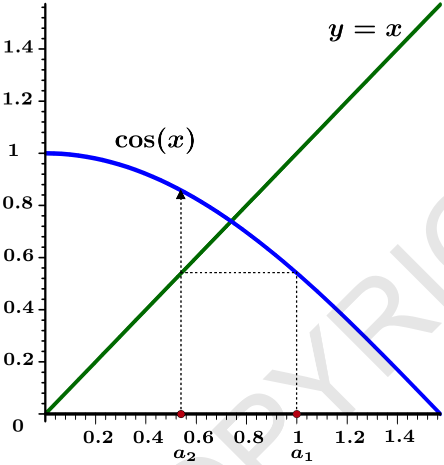
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$

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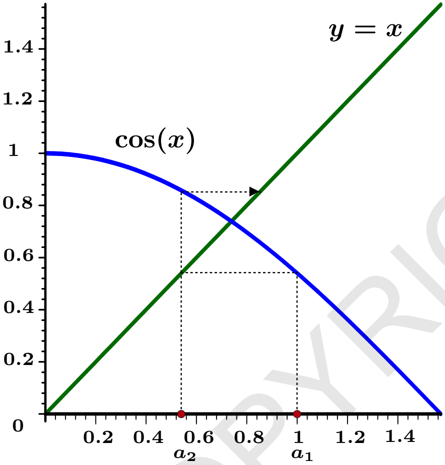
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$

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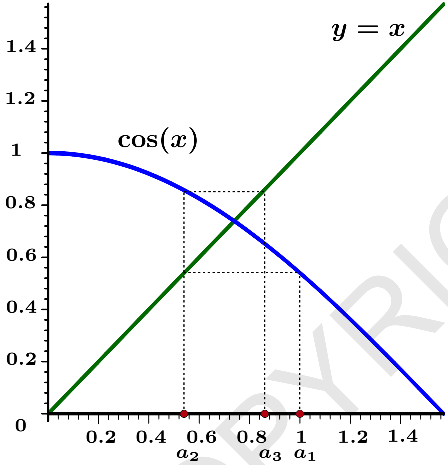
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$

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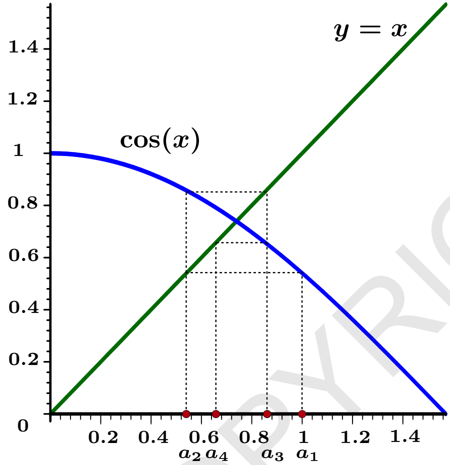
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$

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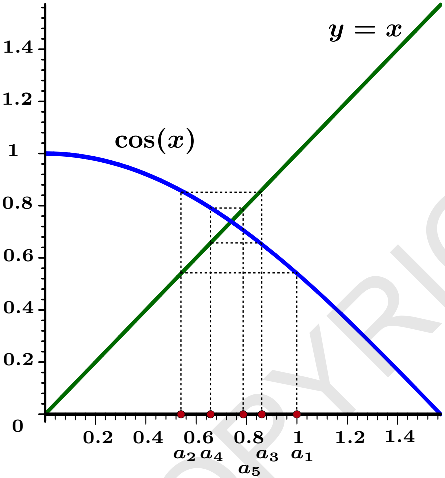
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$
$a_5 = 0.7934803587,$

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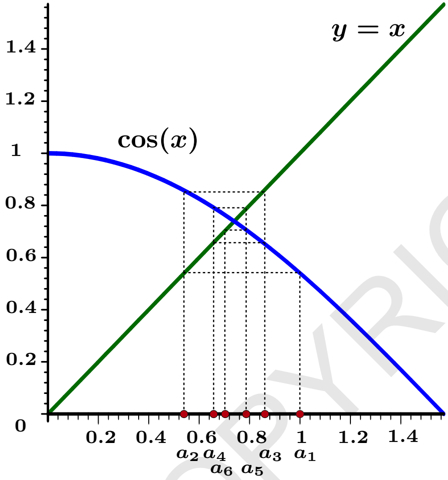
Example 4 (continued)



$$a_1 = 1,$$
$$a_2 = 0.5403023059,$$
$$a_3 = 0.8575532158,$$
$$a_4 = 0.6542897905,$$
$$a_5 = 0.7934803587,$$
$$a_6 = 0.7013687737,$$

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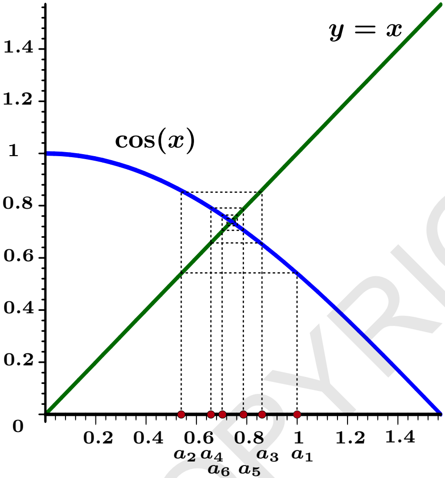
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$
$a_5 = 0.7934803587,$
$a_6 = 0.7013687737,$
$a_7 = 0.7639596829,$
$a_8 = 0.7221024250,$
$a_9 = 0.7504177618,$

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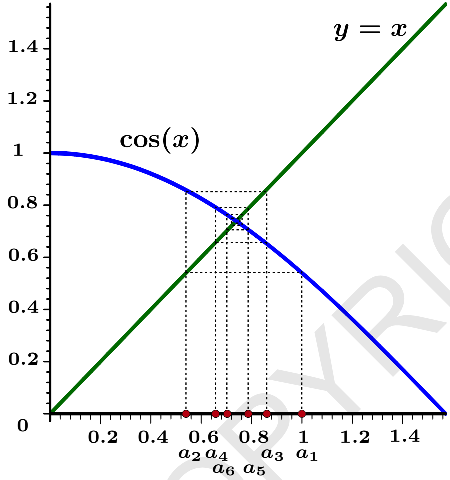
Example 4 (continued)



$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$
$a_5 = 0.7934803587,$
$a_6 = 0.7013687737,$
$a_7 = 0.7639596829,$
$a_8 = 0.7221024250,$
$a_9 = 0.7504177618,$

0.7314040424, 0.7442373549, 0.7356047404, 0.7414250866, 0.7375068905, 0.7401473356, 0.7383692041, 0.7395672022, 0.7387603199, 0.7393038924, 0.7389377567, 0.7391843998, ...

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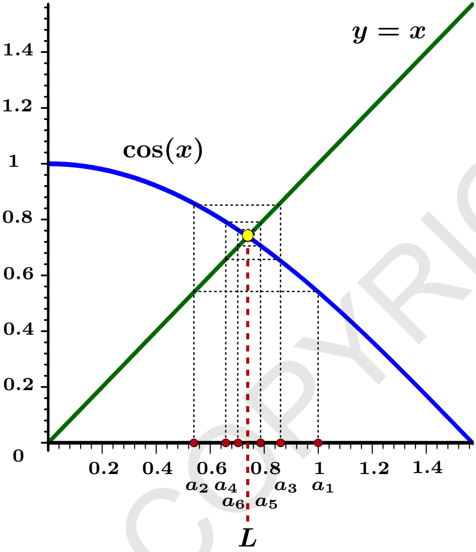
Example 4 (continued)



$a_{72} = 0.7390851332$,
$a_{73} = 0.7390851332$,

and

$a_{74} = 0.7390851332$

suggest that $\{a_n\}$ converges to some L.

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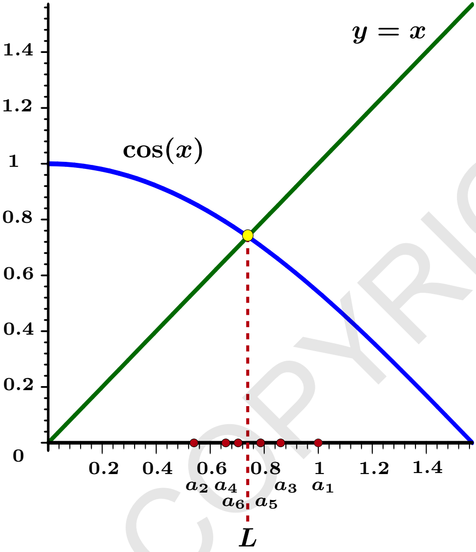
Example 4 (continued)



$a_{72} = 0.7390851332$,
$a_{73} = 0.7390851332$,

and

$a_{74} = 0.7390851332$

suggest that $\{a_n\}$ converges to some L.

In fact,

$$\cos(L) = L.$$

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