Limits of Sequences

Created by

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The Limit of a Sequence

New Heuristic Definition:

We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity if no matter what positive tolerance $\epsilon > 0$ we are given, we can find a cutoff $N \in \mathbb{N}$ such that the terms a_n approximate L with an **error** less than ϵ provided that $n \geq N$.

Formal Definition: [Limit of a Sequence]

We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

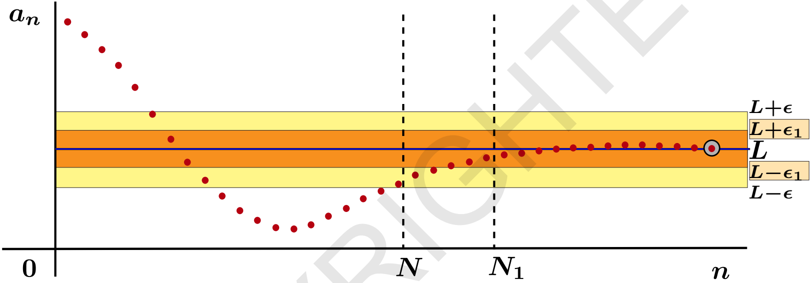
$$\mid a_n - L \mid < \epsilon.$$

In this case, we say that $\{a_n\}$ *converges* to L and we write

$$\lim_{n\to\infty} a_n = L.$$

If no such L exists we say that $\{a_n\}$ *diverges*.

The Limit of a Sequence



1. Identify L.
2. Specify the error $\epsilon > 0$.
3. Find the cutoff N.
4. Choose a smaller ϵ_1.
5. **Repeat Step 3** with a larger N_1.

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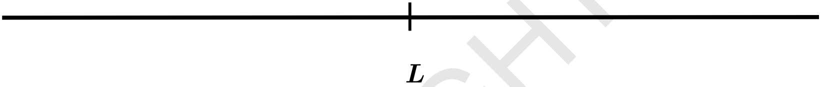

It is useful to look at how this works on the Real line.

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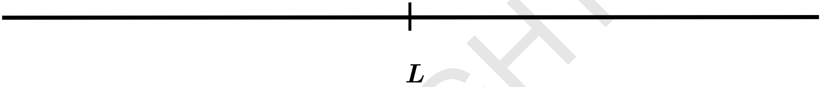

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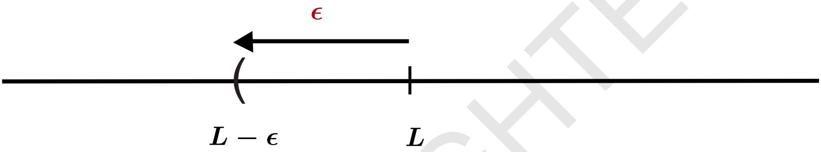

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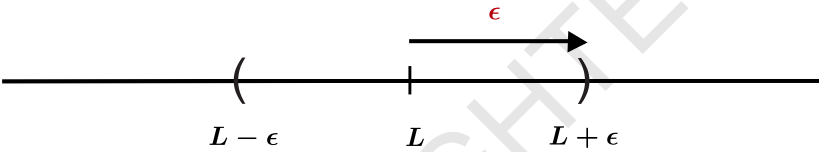

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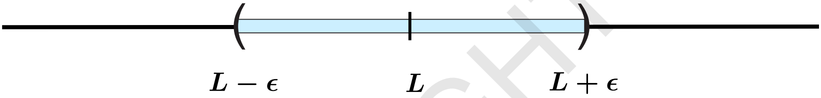

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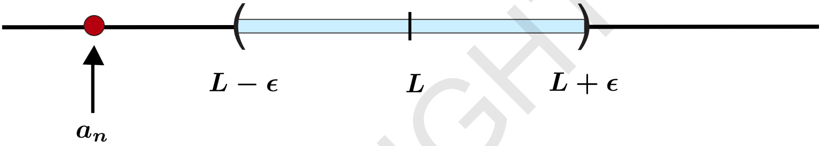

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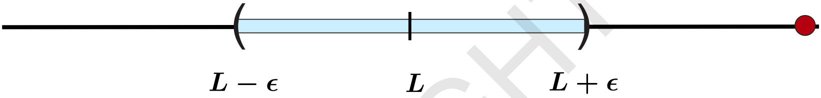

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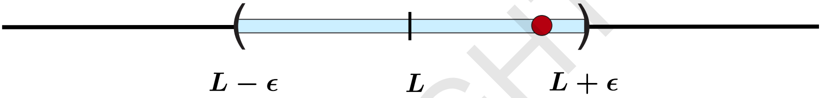

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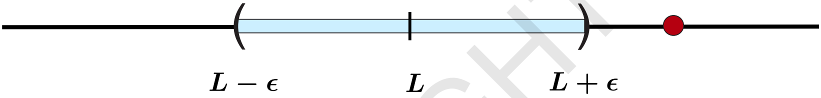

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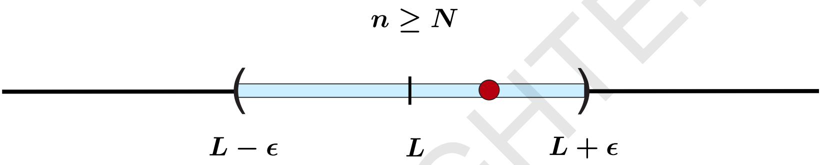

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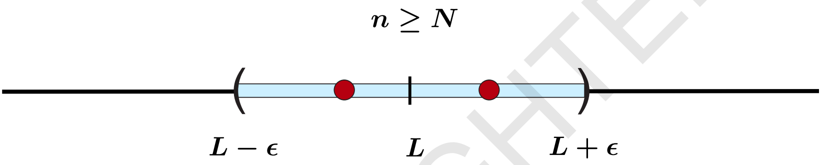

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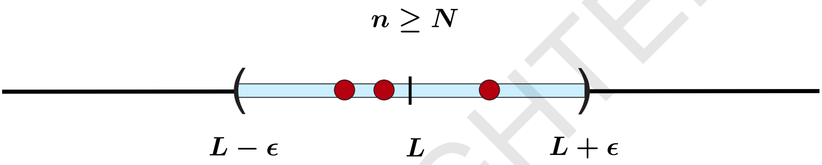

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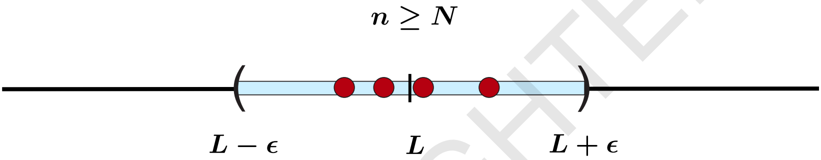

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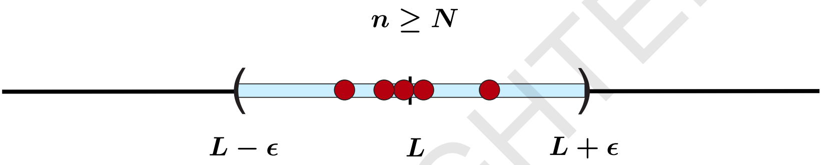

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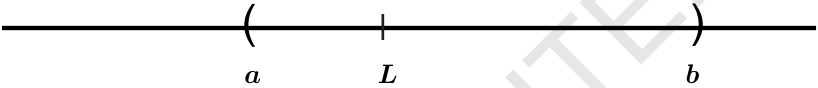

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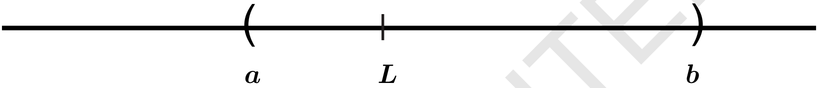

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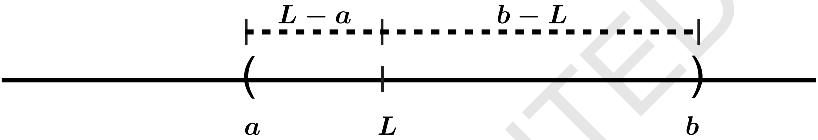

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$$\epsilon \leq \min\{L - a, b - L\}.$$

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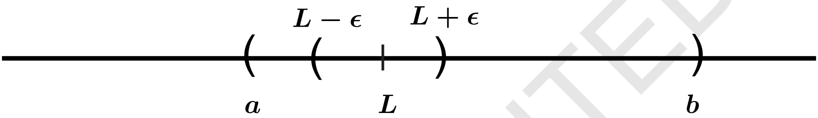

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$$(L-\epsilon, L+\epsilon) \subseteq (a,b).$$

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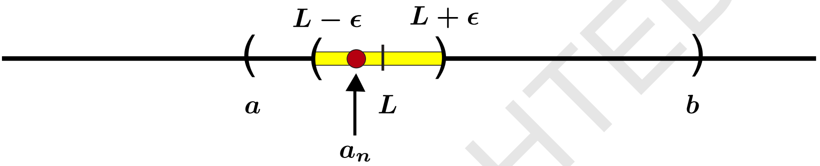

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If n is large enough, then $a_n \in (L-\epsilon, L+\epsilon)$

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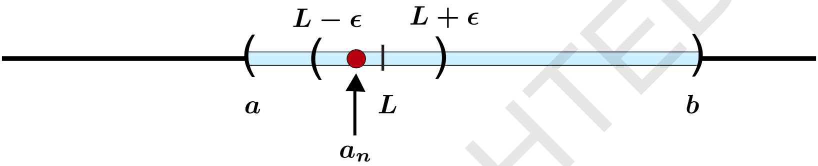

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Then

$$(L-\epsilon, L+\epsilon) \subseteq (a,b).$$

If n is large enough, then $a_n \in (L-\epsilon, L+\epsilon)$ and hence

$$a_n \in (a,b).$$

Summary

Theorem

The following statements are equivalent:

1. $\lim_{n\to\infty} a_n = L$.
2. Every interval $(L-\epsilon, L+\epsilon)$ contains a **tail** of $\{a_n\}$.
3. Every interval $(L-\epsilon, L+\epsilon)$ contains **all but finitely many terms** of $\{a_n\}$.
4. Every interval (a, b) containing L contains a **tail** of $\{a_n\}$.
5. Every interval (a, b) containing L contains **all but finitely many terms** of $\{a_n\}$.

Important Note: Changing finitely many terms in $\{a_n\}$ does not affect convergence.

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