Absolute Value and the Triangle Inequality

Created by

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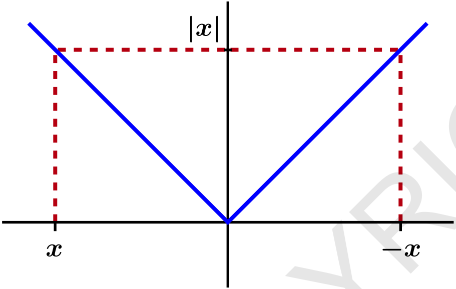
Absolute Value



Definition: [Absolute Value]

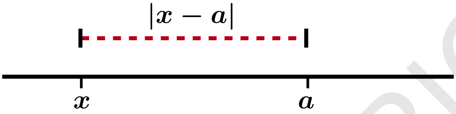
The *absolute value* of a real number x is the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Properties:

1) $|x| \geq 0$
2) $|x| = |-x|$

Geometric Interpretation of Absolute Value



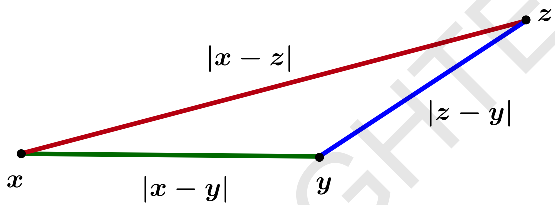
Remark: $|x| = \sqrt{x^2}$

The absolute value is the one-dimensional analogue of $\sqrt{x^2 + y^2}$, which measures the "length" of a vector (x, y) in the plane.

Geometric Interpretation:

- $| x |=$ the distance from x to 0.
- $| x - a |=$ the distance from x to a.

Triangle Inequality



Theorem: [Triangle Inequality]

For any $x, y, z \in \mathbb{R}$

$$|\, x - y \,| \;\leq |\, x - z \,| + |\, z - y \,|$$

Remark: The length of any side of a triangle is less than or equal to the sum of the other two sides.

Proof of the Triangle Inequality

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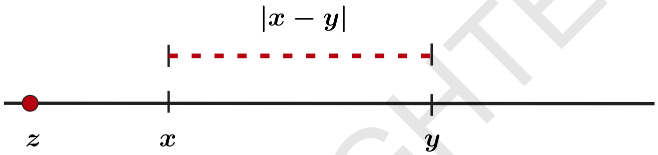

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We may assume $x < y$. There are three cases:

Case 1: $z < x$

Then

$$|\ x - y\ |$$

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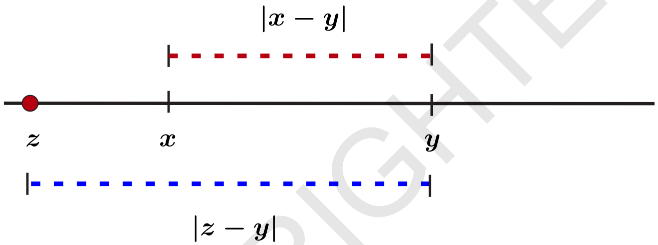

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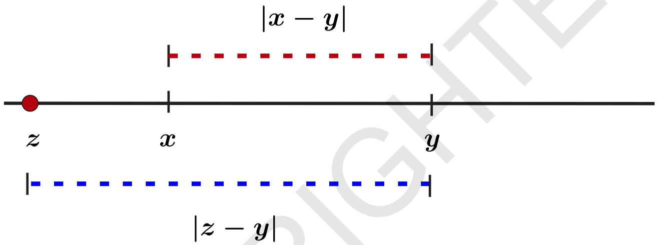

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Case 2: $x \leq z \leq y$

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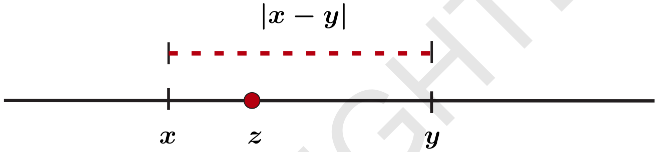

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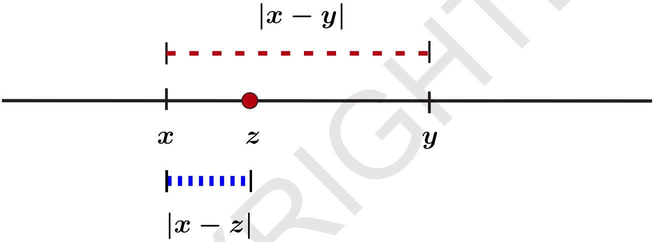
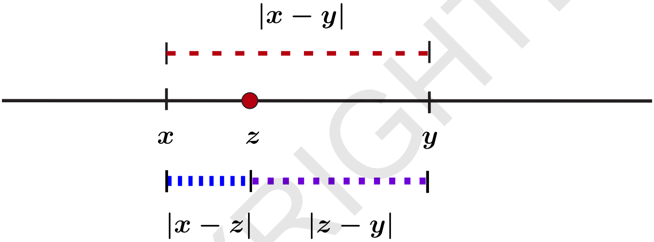


Case 2: $x \leq z \leq y$

Then

$$|\, x - y \,| \;= |\, x - z \,|$$

Proof of the Triangle Inequality



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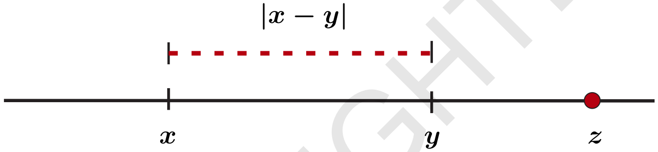

Case 3: $y < z$

Then

$$|\, x - y \,|$$

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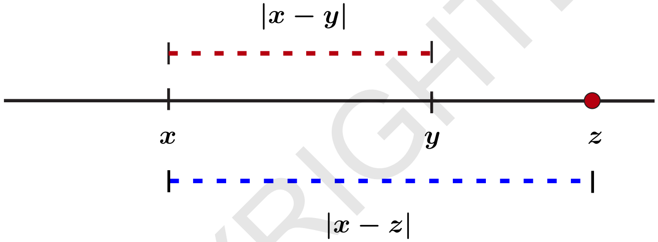

Case 3: $y < z$

Then

$$|x - y| < |x - z|$$

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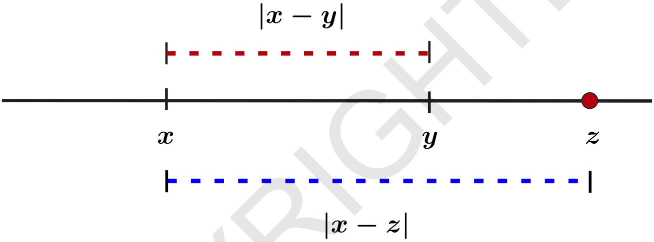


Case 3: $y < z$

Then

$$|x - y| < |x - z| \leq |x - z| + |z - y|$$

Variant of the Triangle Inequality

Theorem: [Triangle Inequality II]

Let $x, y \in \mathbb{R}$. Then

$$|\, x + y \,| \leq |\, x \,| + |\, y \,|$$

Proof: Let $x, y \in \mathbb{R}$. Applying the Triangle Inequality to $x, -y$ and $z = 0$ gives

$$\begin{aligned} |\, x + y \,| &= |\, x - (-y) \,| \\ &\leq |\, x - 0 \,| + |\, 0 - (-y) \,| \\ &= |\, x \,| + |\, y \,| \end{aligned}$$

◀

Example

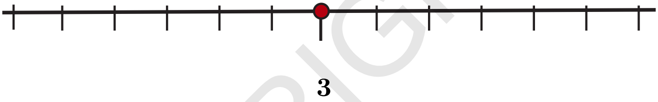
Problem: Find all $x \in \mathbb{R}$ such that $\mid x - 3 \mid < 2$.

Approach 1: Algebraic Solution

$$\mid x - 3 \mid < 2 \iff -2 < x - 3 < 2 \iff -2 + 3 < x < 2 + 3$$

Solution: $x \in (1, 5)$. ◀

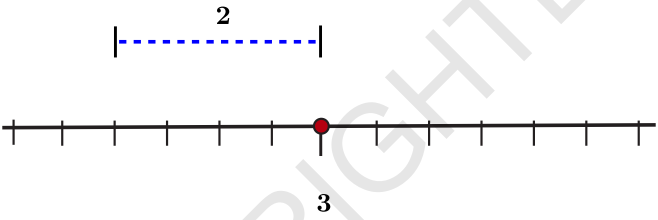
Example



$$|x - 3| < 2$$

Approach 2: Geometric Solution

Example



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Approach 2: Geometric Solution

"distance from x to 3 is less than 2"

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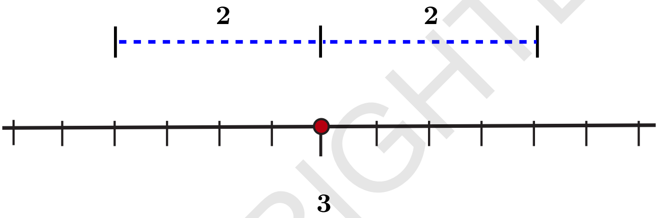

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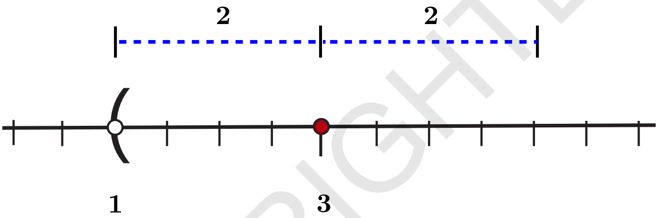
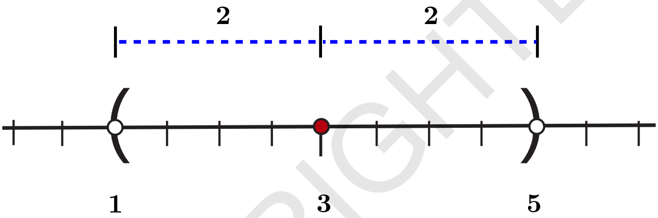


$$|x - 3| < 2$$

Approach 2: Geometric Solution

"distance from x to $\mathbf{3}$ is less than $\mathbf{2}$" $\Longrightarrow x > \mathbf{1}$

Example

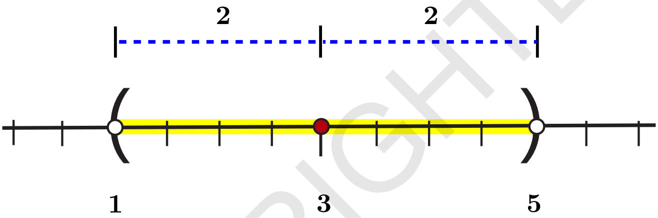


$$|x - 3| < 2$$

Approach 2: Geometric Solution

"distance from x to $\mathbf{3}$ is less than $\mathbf{2}$" $\implies x > 1$ and $x < 5$.

Example



$$|x - 3| < 2$$

Approach 2: Geometric Solution

"distance from x to 3 is less than 2" $\implies x > 1$ and $x < 5$.

Solution: $x \in (1, 5)$.

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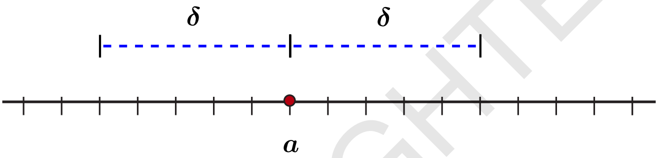

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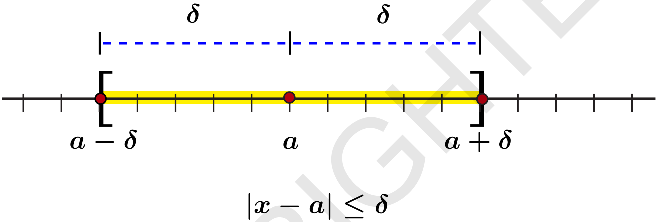

Important Inequalities: Let $\delta > 0$,

1. $|x - a| \leq \delta$ if and only if $x \in [a - \delta, a + \delta]$.

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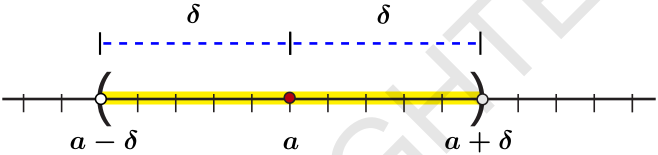
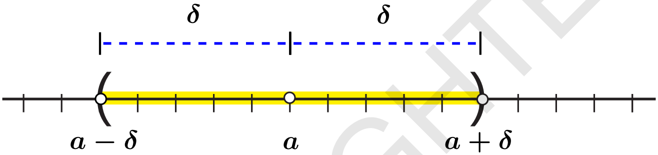


$$|x - a| < \delta$$

Important Inequalities: Let $\delta > 0$,

1. $|x - a| \leq \delta$ if and only if $x \in [a - \delta, a + \delta]$.
2. $|x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta)$.

Important Inequalities



$$\boldsymbol{0 < |x - a| < \delta}$$

Important Inequalities: Let $\delta > 0$,

1. $|x - a| \leq \delta$ if and only if $x \in [a - \delta, a + \delta]$.
2. $|x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta)$.
3. $0 < |x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta) \setminus \{a\}$.