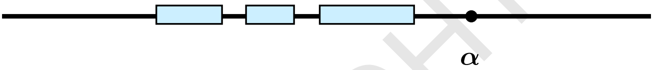
Least Upper Bound Property

Created by

Barbara Forrest and Brian Forrest

Bounded Sets

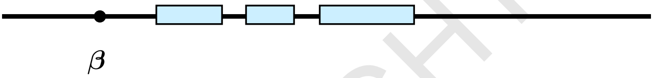


Definition: [Upper/Lower Bounds]

Let $S \subset \mathbb{R}$.

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.

Bounded Sets

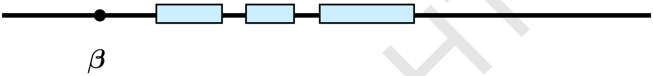


Definition: [Upper/Lower Bounds]

Let $S \subset \mathbb{R}$.

1. α is an *upper bound* for S if $x \leq \alpha$ for all $x \in S$.
2. β is a *lower bound* for S if $\beta \leq x$ for all $x \in S$.

Bounded Sets

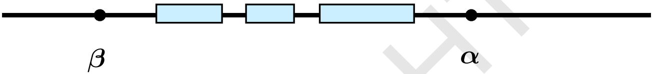


Definition: [Bounded Sets]

Let $S \subset \mathbb{R}$.

1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has a lower bound β.

Bounded Sets

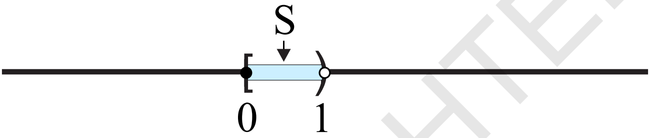


Definition: [Bounded Sets]

Let $S \subset \mathbb{R}$.

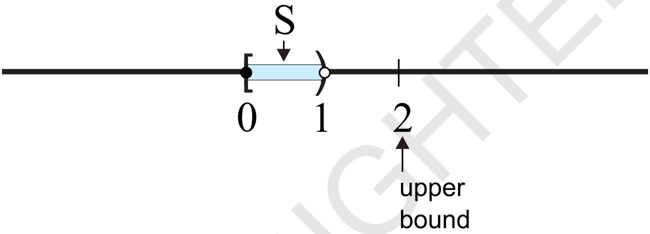
1. S is *bounded above* if S has an upper bound α.
2. S is *bounded below* if S has a lower bound β.
3. S is *bounded* if S is bound above and bounded below.

The Set $[0, 1)$



Let $S = [0, 1)$.

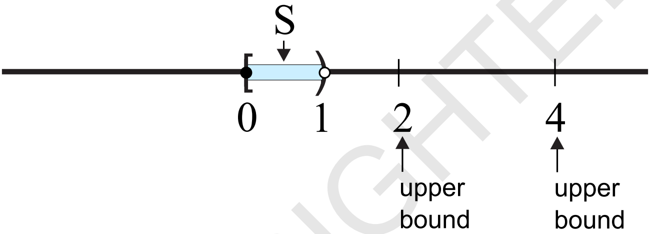
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.

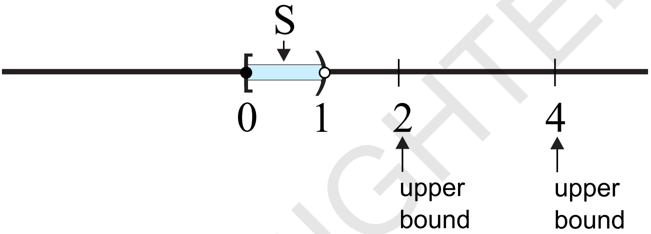
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.

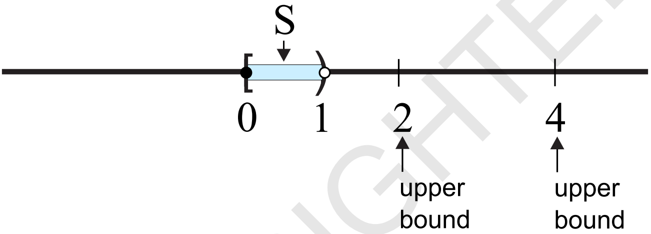
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.

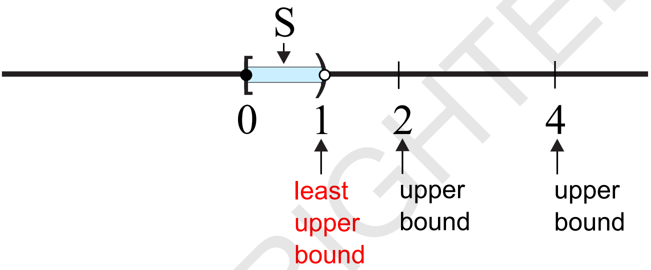
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.
- 1 is a *special* upper bound for S.

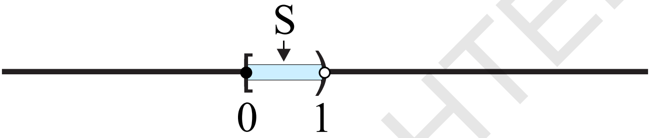
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded above by 2.
- S is also bounded above by 4.
- S has infinitely many upper bounds.
- 1 is a *special* upper bound for S.
- 1 is the smallest or *least upper bound* for S.

The Set $[0, 1)$



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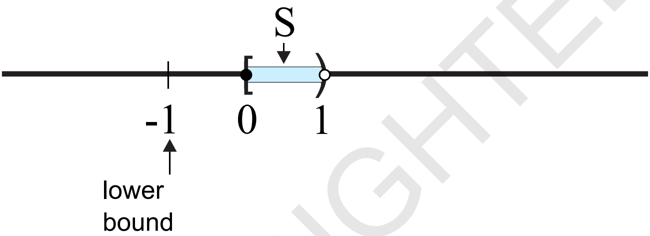
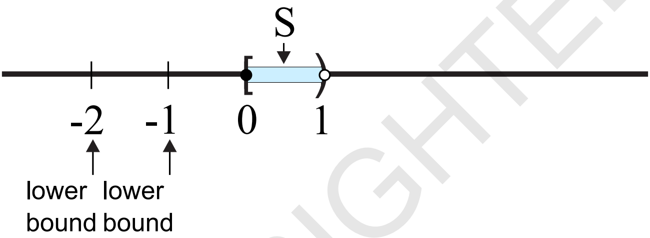
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.

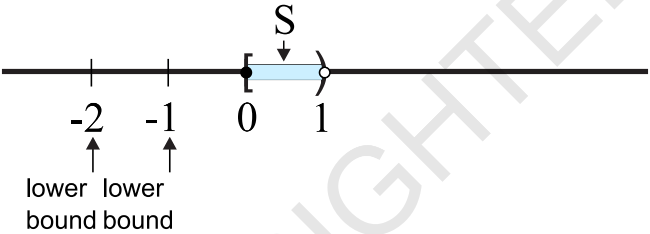
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.

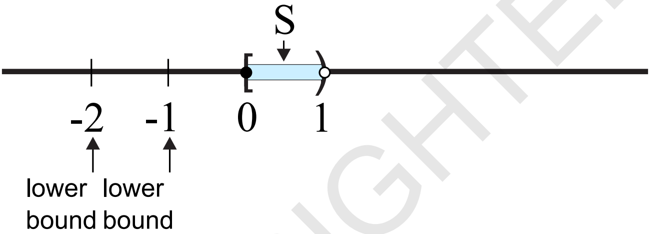
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.

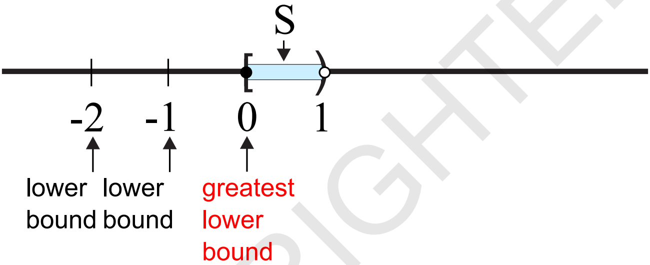
The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.
- 0 is a *special* lower bound for S.

The Set $[0, 1)$



Let $S = [0, 1)$.

- S is bounded below by -1.
- S is also bounded below by -2.
- S has infinitely many lower bounds.
- 0 is a *special* lower bound for S.
- 0 is the largest or *greatest lower bound* for S.

Least Upper Bound

Definition: [Least Upper Bound]

We say that $\alpha \in \mathbb{R}$ is the *least upper bound* for a set $S \subset \mathbb{R}$ if

1. α is an upper bound for S, and
2. if γ is an upper bound for S, then $\alpha \leq \gamma$.

If a set S has a least upper bound, then we denote it by $lub(S)$.

The least upper bound of S is often called the *supremum* of S, denoted by $sup(S)$.

Greatest Lower Bound

Definition: [Greatest Lower Bound]

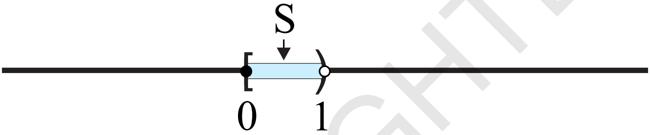
We say that $\beta \in \mathbb{R}$ is the *greatest lower bound* for a set $S \subset \mathbb{R}$ if

1. β is a lower bound for S, and
2. if γ is a lower bound for S, then $\beta \geq \gamma$.

If a set S has a greatest bound, then we denote it by $glb(S)$.

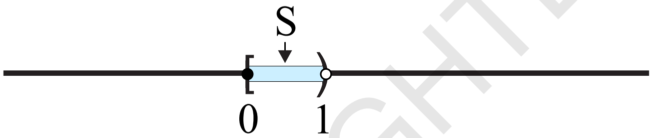
The greatest lower bound of S is often called the *infimum* of S, denoted by $inf(S)$.

The Set $[0, 1)$



Let $S = [0, 1)$.

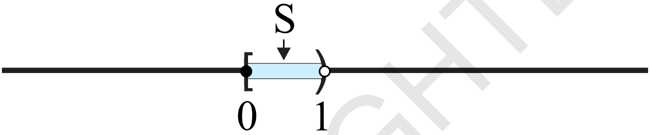
The Set $[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$

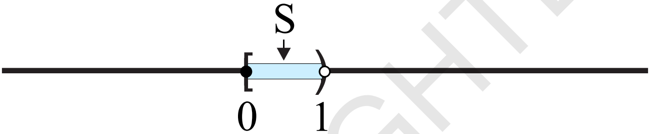
The Set $[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$
- $glb(S) = 0$

The Set $[0, 1)$

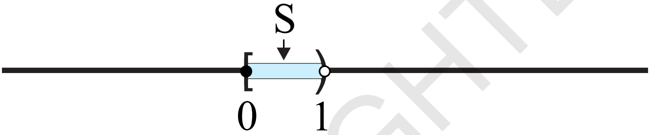


Let $S = [0, 1)$.

- $lub(S) = 1$
- $glb(S) = 0$

Note: $glb(S) = 0 \in S$,

The Set $[0, 1)$



Let $S = [0, 1)$.

- $lub(S) = 1$
- $glb(S) = 0$

Note: $glb(S) = 0 \in S$, but $lub(S) = 1 \notin S$.

Maximum and Minimum

Definition: [Maximum/Mininum]

1. If S contains $\alpha = lub(S)$, then α is called the *maximum* of S and is denoted by $max(S)$.
2. If S contains $\beta = glb(S)$, then β is called the *minimum* of S and is denoted by $min(S)$.

Example: If S is a finite set with n elements

$$S = \{a_1 < a_2 < \cdots < a_n\},$$

then

- $a_n = lub(S) = max(S)$, and
- $a_1 = glb(S) = min(S)$.

The Empty Set

Problem: Does every set S that is bounded above have a LUB?

Axiom: [Least Upper Bound Property or LUBP]

A **nonempty** subset $S \subset \mathbb{R}$ that is bounded above always has a least upper bound.

Example

Example: Let S be the terms in the sequence $\{1 - \frac{1}{n}\}$.

That is,

$$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}.$$

Note: Each term is less than 1, but we can get as close to 1 as we would like so long as the index n is large enough.

Hence,

$$1 = lub(S).$$

We also know that

$$1 = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right).$$

The fact that the limit and the least upper bound agree is no accident.