Examples of Recursively Defined Sequences

Created by

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Fibonacci Sequence

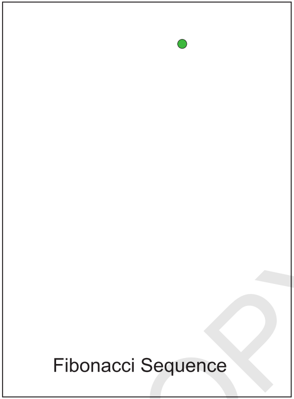
Recall: In a recursively defined sequence, the value of a term a_n is determined by a formula that involves terms from earlier in the sequence.

Problem: (Fibonacci - *Liber Abaci 1202*) Assume that

- A newly born pair of breeding rabbits can mate after one month.
- Each female in a breading pair will produce exactly one additional breeding pair one month later and once a new pair is born the existing pair will immediately breed.
- The rabbits never die.

Let F_n denote the number of rabbit pairs at the beginning of month n. Assuming that we begin with one newly born pair of rabbits, find a recursive formula for F_n.

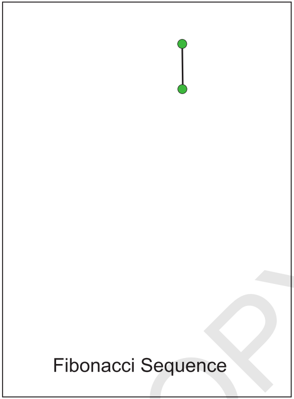
Fibonacci Sequence



Solution:

- $F_1 = 1,$

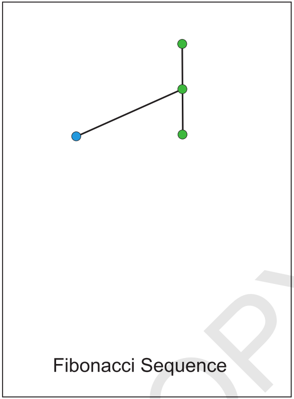
Fibonacci Sequence



Solution:

- $F_1 = 1, \quad F_2 = 1$

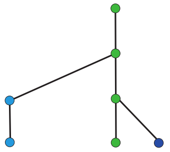
Fibonacci Sequence



Solution:

- $F_1 = 1, \quad F_2 = 1$
- $F_3 = F_2 + F_1 = 1 + 1 = 2$

Fibonacci Sequence

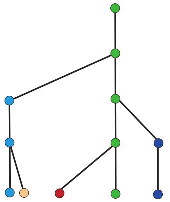


Fibonacci Sequence

Solution:

- $F_1 = 1, \quad F_2 = 1$
- $F_3 = F_2 + F_1 = 1 + 1 = 2$
- $F_4 = F_3 + F_2 = 2 + 1 = 3$

Fibonacci Sequence



Fibonacci Sequence

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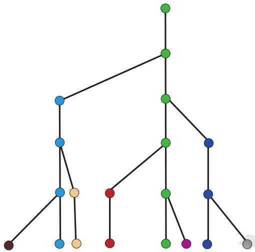
- $F_5 = F_4 + F_3 = 3 + 2 = 5$

Fibonacci Sequence



Fibonacci Sequence

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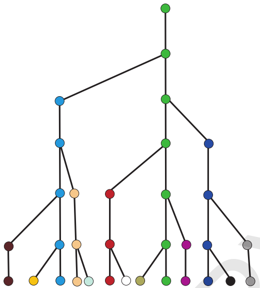
- $F_6 = F_5 + F_4 = 5 + 3 = 8$

Fibonacci Sequence



Fibonacci Sequence

Solution:

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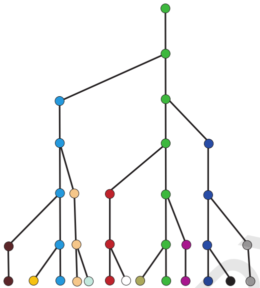
- $F_7 = F_6 + F_5 = 8 + 5 = 13$

Fibonacci Sequence



Fibonacci Sequence

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- $F_6 = F_5 + F_4 = 5 + 3 = 8$
- $F_7 = F_6 + F_5 = 8 + 5 = 13$

Then

$$F_n = F_{n-1} + F_{n-2}.$$

Fibonacci Sequence

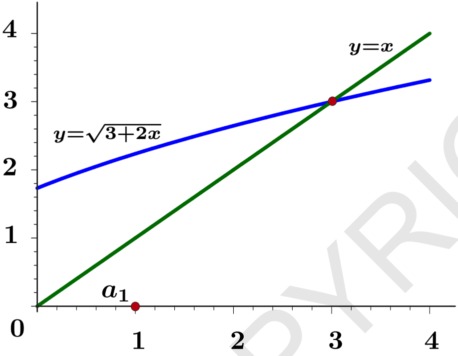
Definition: [Fibonacci Sequence]

The recursively defined sequence

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

is called the *Fibonacci Sequence*.

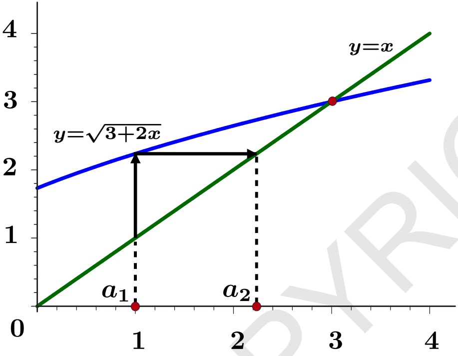
The Sequence $a_1 = 1$, $a_{n+1} = \sqrt{3 + 2a_n}$



Example: Let $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$.

n	a_n
1	1

The Sequence $a_1 = 1$, $a_{n+1} = \sqrt{3 + 2a_n}$



Example: Let $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$.

n	a_n
1	1
2	2.236067977

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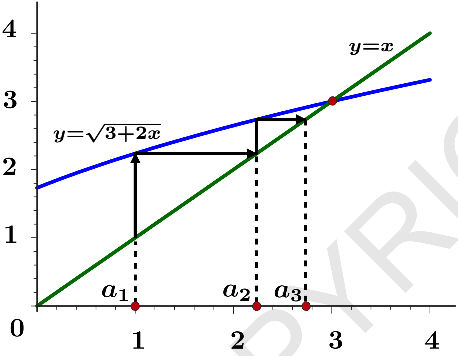

Example: Let $a_1 = 1$ and

$a_{n+1} = \sqrt{3 + 2a_n}$.

n	a_n
1	1
2	2.236067977
3	2.733520798

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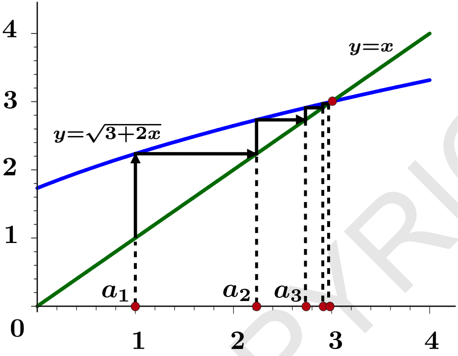

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n	a_n
1	1
2	2.236067977
3	2.733520798
4	2.909818138
5	2.969787244
6	2.989912121
7	2.996635487
8	2.998878286
9	2.999626072
10	2.999875355

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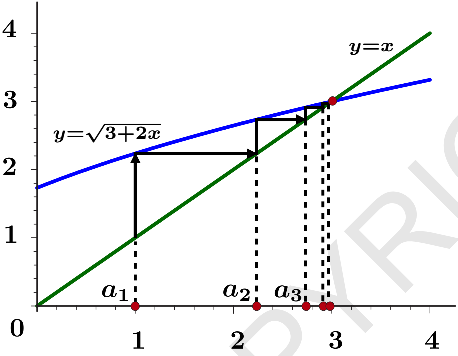

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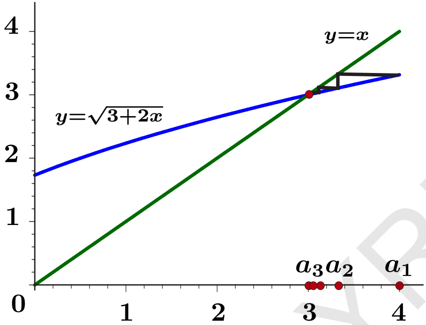
Note: $a_n \cong 3$ for large n.

The Sequence $a_1 = 4$, $a_{n+1} = \sqrt{3 + 2a_n}$



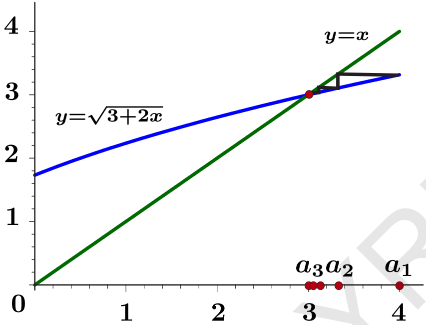
Example: Let $a_1 = 4$ and

$a_{n+1} = \sqrt{3 + 2a_n}$.

n	a_n	a_n
1	4	
2	3.31662479	
3	3.103747667	
4	3.034385495	
5	3.011440019	
6	3.003810919	
7	3.001270038	
8	3.000423316	
9	3.000141102	
10	3.000047034	

Question: What if $a_1 = 4$?

The Sequence $a_1 = 4$, $a_{n+1} = \sqrt{3 + 2a_n}$



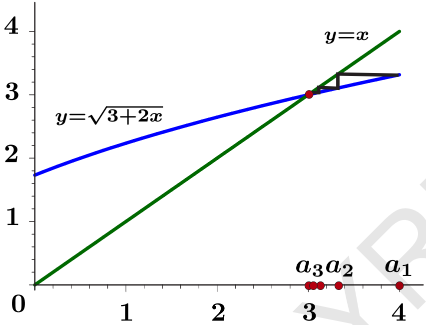
Example: Let $a_1 = 4$ and

$$a_{n+1} = \sqrt{3 + 2a_n}.$$

n	a_n	a_n
1	4	1756
2	3.31662479	59.28743543
3	3.103747667	11.02609953
4	3.034385495	5.005217184
5	3.011440019	3.606997972
6	3.003810919	3.195934283
7	3.001270038	3.064615566
8	3.000423316	3.021461754
9	3.000141102	3.007145409
10	3.000047034	3.002380858

Question: What if $a_1 = 4$? Or if $a_1 = 1756$?

The Sequence $a_1 = 4$, $a_{n+1} = \sqrt{3 + 2a_n}$



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1	4	1756
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Question: What if $a_1 = 4$? Or if $a_1 = 1756$?

Problem: What happens if $a_1 = 3$?

Heron's Algorithm for Finding Square Roots

Example: Calculate $\sqrt{479678}$.

n	a_n	a_n
1	1	700
2	239839.5	692.6271429
3	119920.75	692.5879017
4	59962.37498	692.5879006
5	29985.18731	692.5879006
6	15000.59224	692.5879006
7	7516.284755	692.5879006
8	3790.051626	692.5879006
9	1958.307009	692.5879006
10	1101.62613	692.5879006
11	768.5266734	692.5879006
12	696.3396879	692.5879006
13	692.5980076	692.5879006
14	692.5879006	692.5879006

Solution:
By using *Heron's Algorithm* with $\alpha = 479678$, $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}(a_n + \frac{\alpha}{a_n})$$

the sequence of values listed in the table is generated. Thus,

$$\sqrt{479678} \approx 692.5879006$$

Remark:
The algorithm works with any $a_1 > 0$, but performance improves if a_1 is chosen close to $\sqrt{\alpha}$.