Introduction to Sequences

Created by

Barbara Forrest and Brian Forrest

What is a Sequence?

Informally, a sequence is an ordered list of real numbers.

For example a phone number 519-555-1234 is a *finite* sequence of 1-digit numbers

$$\mathbf{5-1-9-5-5-5-1-2-3-4.}$$

Note: Order is important!

During our discussions, sequences will be *infinite ordered lists of real numbers* of the form $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.

What is a Sequence?

More formally:

Definition: [Sequence]

A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

For $n \in \mathbb{N}$, we call $f(n)$ the *nth term* of the sequence and n is called *the index*.

Notation: We often write a_n or b_n or x_n instead of $f(n)$ to denote the nth term.

Specifying a Sequence

A sequence can be specified by:

1) Providing the explicit function:

- $f(n) = \frac{1}{n}$
- $a_n = \frac{1}{n}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
- $\{\frac{1}{n}\}_{n=1}^{\infty}$ or simply $\{\frac{1}{n}\}$

Note: Some references use round brackets instead of curly brackets to denote sequences. As such, you may encounter a sequence written as $(a_1, a_2, a_3, \cdots, a_n, \cdots)$ or $(a_n)_{n=1}^{\infty}$ or (a_n).

Specifying a Sequence

2) Recursion:

Future terms in a sequence are defined from existing terms.

Example: Let $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.

What is a_5?

- $a_1 = 1$
- $a_2 = \frac{1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$
- $a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$
- $a_4 = \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$
- $a_5 = \frac{1}{1+a_4} = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$

Exercise: Can you find an explicit formula for a_n?

Subsequences: New Sequences from Old

Definition: [Subsequence]

Given a sequence $\{a_n\}$ and a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$n_1 < n_2 < n_3 \ldots < n_k \ldots$$

define

$$b_k = a_{n_k}.$$

The sequence $\{b_k\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_k}, \ldots\}$ is called a *subsequence* of $\{a_n\}$.

Example: Let $a_n = \frac{1}{n}$ and $n_k = 2k$. Then

$$b_k = a_{n_k} = a_{2k} = \frac{1}{2k}.$$

Subsequences: New Sequences from Old

$$\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots, a_{2k}, \ldots\}$$

$$\{b_k\} = \{b_1, b_2, \ldots, b_k, \ldots\}$$

$$= \{a_2, a_4, \ldots, a_{2k}, \ldots\}$$

Subsequences: New Sequences from Old

Example: Given $\{a_n\}$ and $j \in \mathbb{N} \cup \{0\}$, let

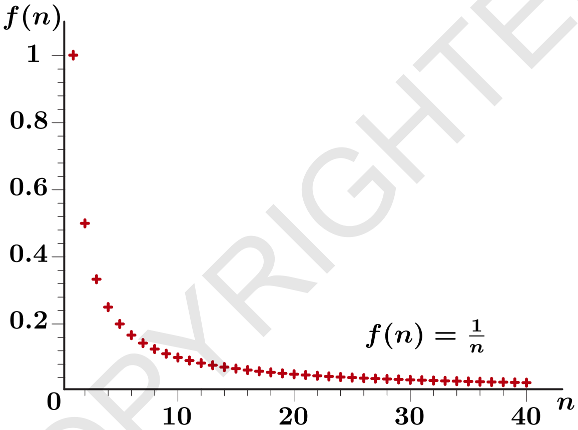
$$\{b_k\} = \{a_{k+j}\} = \{a_{1+j}, a_{2+j}, a_{3+j}, \ldots\}.$$

For example if $j = 3$, we get

$$\{a_{k+3}\}_{k=1}^{\infty} = \{a_{1+3}, a_{2+3}, a_{3+3}, \ldots\} = \{a_4, a_5, a_6, \ldots\}.$$

Note: Such a subsequence is called a *tail* of $\{a_n\}$.

Graphical Representation of a Sequence



1) Standard 2-dimensional plot.

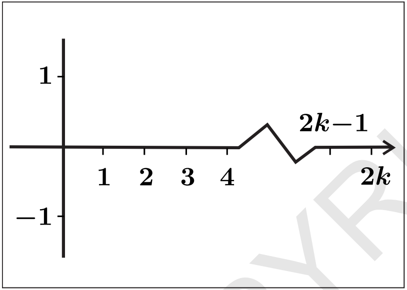
Example: $\{\frac{1}{n}\}$.

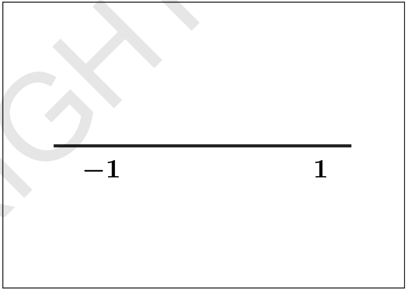
Graphical Representation of a Sequence

2) 1-dimensional plot.

Example: $\{\frac{1}{n}\}$.

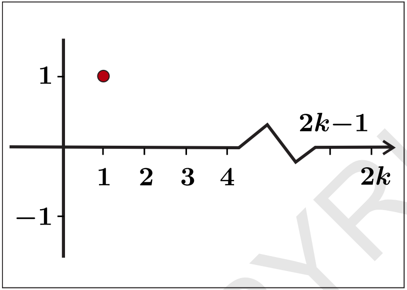
Graphical Representation of a Sequence

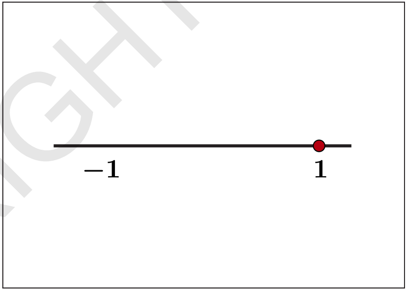




Example: $a_n = (-1)^{n+1}$

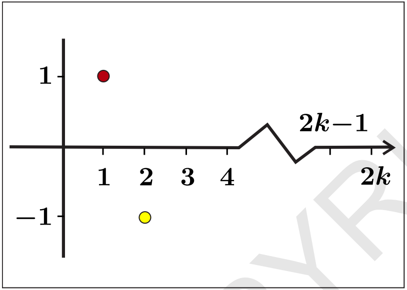
Graphical Representation of a Sequence

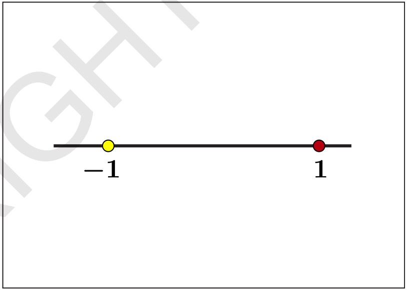




Example: $a_n = (-1)^{n+1} \Rightarrow \{1,$

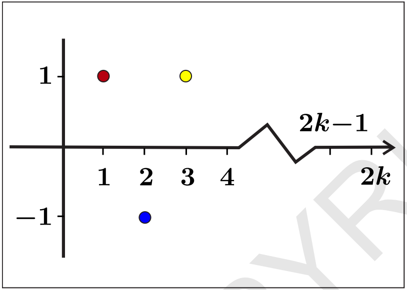
Graphical Representation of a Sequence

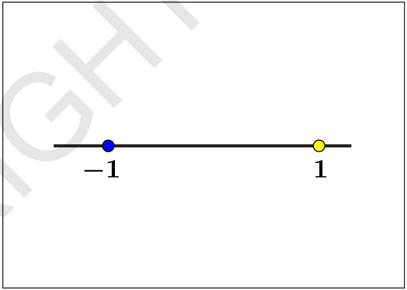




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1,$

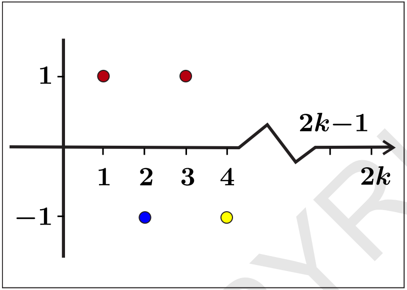
Graphical Representation of a Sequence

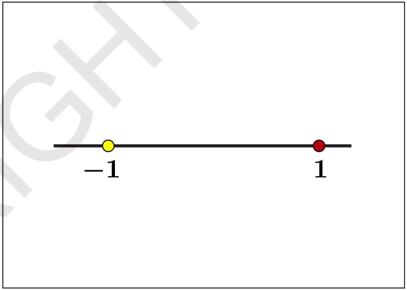




Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1,$

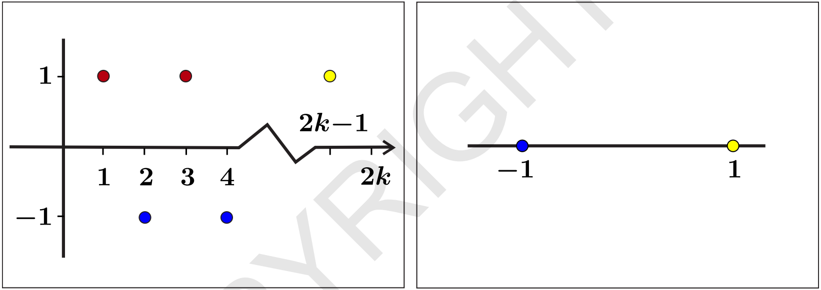
Graphical Representation of a Sequence





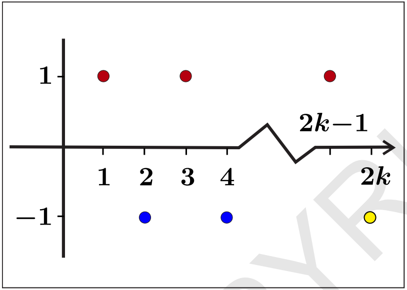
Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1,$

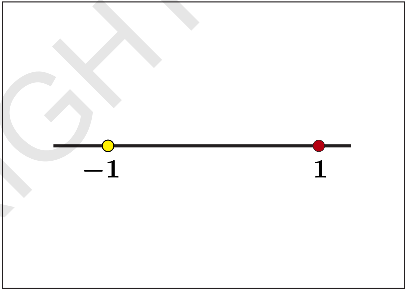
Graphical Representation of a Sequence



Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1,$

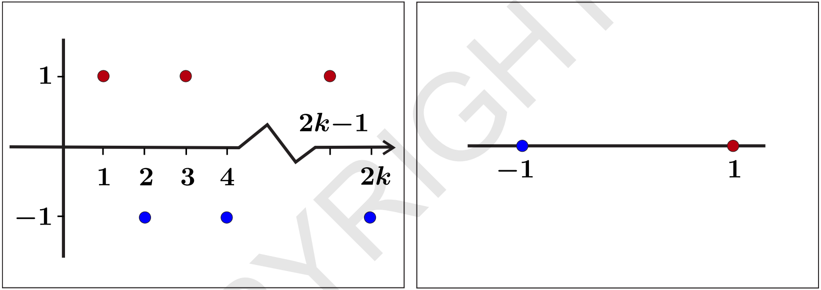
Graphical Representation of a Sequence





Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1, -1,$

Graphical Representation of a Sequence



Example: $a_n = (-1)^{n+1} \Rightarrow \{1, -1, 1, -1, \ldots 1, -1, \ldots\}$