Introduction to Series

Created by

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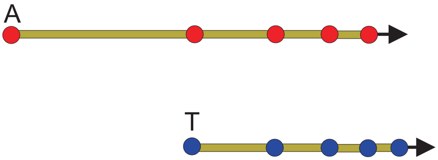
A Problem to Consider

Problem:

Can infinitely many tasks be performed in a finite amount of time?

The Paradox of Achilles



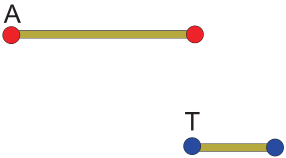
Paradox of Achilles

- Achilles races a tortoise who is given a head start.
- Achilles **reaches the point where the tortoise began**, but the tortoise has moved ahead to a new point.
- Achilles **reaches the new point**. Again, the tortoise has moved ahead.
- Achilles reaches the next point, and again the tortoise has moved ahead.
- And so on . . ..
- **Conclusion:** Achilles can never catch the tortoise!!!

Resolving The Paradox of Achilles

Resolving the Paradox

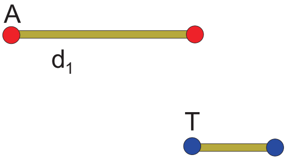
Resolving The Paradox of Achilles



Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

Resolving The Paradox of Achilles



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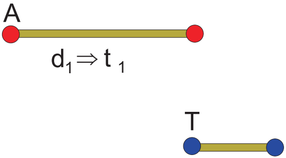
- $d_1 =$ distance Achilles traveled in stage 1

Resolving The Paradox of Achilles

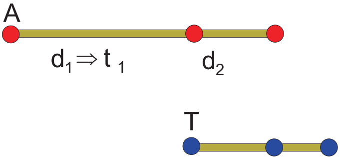


Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

- d_1 = distance Achilles traveled in stage 1 $\Rightarrow$ t_1 = time to complete stage 1

Resolving The Paradox of Achilles

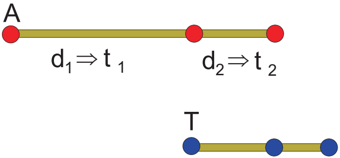


Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

- $d_1 =$ distance Achilles traveled in stage 1 $\Rightarrow t_1 =$ time to complete stage 1
- $d_2 =$ distance Achilles traveled in stage 2

Resolving The Paradox of Achilles

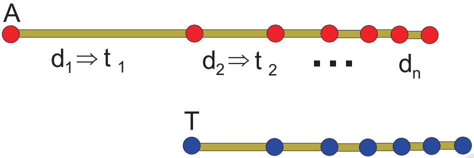


Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

- $d_1 =$ distance Achilles traveled in stage 1 $\Rightarrow t_1 =$ time to complete stage 1
- $d_2 =$ distance Achilles traveled in stage 2 $\Rightarrow t_2 =$ time to complete stage 2

Resolving The Paradox of Achilles

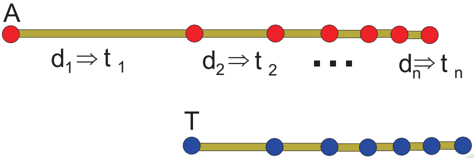


Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

- d_1 = distance Achilles traveled in stage 1 $\Rightarrow t_1$ = time to complete stage 1
- d_2 = distance Achilles traveled in stage 2 $\Rightarrow t_2$ = time to complete stage 2
- d_n = distance Achilles traveled in stage n

Resolving The Paradox of Achilles



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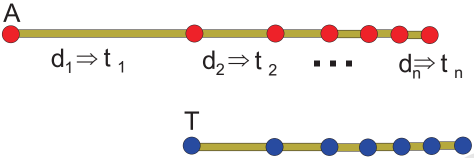
Resolving The Paradox of Achilles



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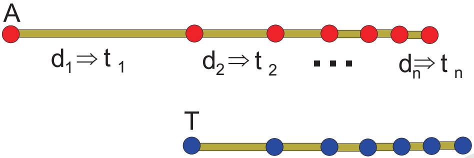
- d_n = distance Achilles traveled in stage n $\Rightarrow t_n$ = time to complete stage n

$$\text{Time to catch the Tortoise} \quad = \quad t_1 + t_2 + \cdots + t_n + \cdots$$

Resolving The Paradox of Achilles



Resolving the Paradox

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- d_2 = distance Achilles traveled in stage 2 $\Rightarrow t_2$ = time to complete stage 2
- d_n = distance Achilles traveled in stage n $\Rightarrow t_n$ = time to complete stage n

$$\begin{aligned} \text{Time to catch the Tortoise} &= t_1 + t_2 + \cdots + t_n + \cdots \\ &= \infty \text{ ?} \end{aligned}$$

Introduction to Series

Problem:

Can we add infinitely many numbers at the same time?

More precisely, given a sequence $\{a_n\}$, we can form the *formal sum*

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

which is called a *series*.

Question:

What does this formal sum represent? Does it have a value?

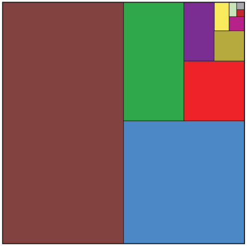
Introduction to Series

Example: What is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\cdots$$

$$=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots?$$

Introduction to Series



Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\cdots$$

$$=1?$$

Introduction to Series

Introduction to Series

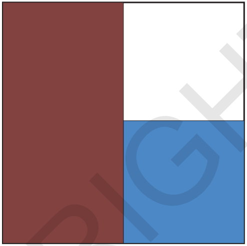
Sum of Areas

$\frac{1}{2}$

Total Area Covered

$1 - \frac{1}{2}$

Introduction to Series



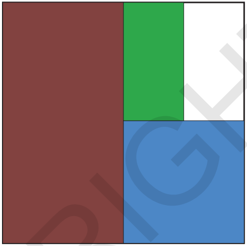
Sum of Areas

$\frac{1}{2} + \frac{1}{4}$

Total Area Covered

$1 - \frac{1}{4}$

Introduction to Series



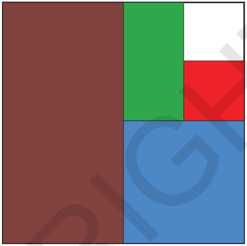
Sum of Areas

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

Total Area Covered

$1 - \frac{1}{8}$

Introduction to Series



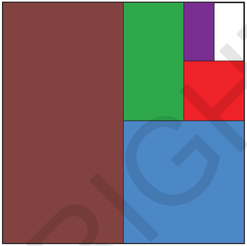
Sum of Areas

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

Total Area Covered

$1 - \frac{1}{16}$

Introduction to Series



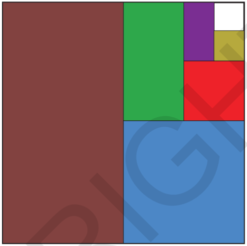
Sum of Areas

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$$

Total Area Covered

$$1-\frac{1}{32}$$

Introduction to Series



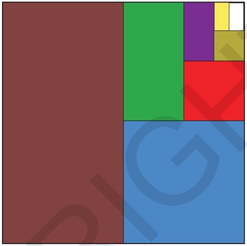
Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

Total Area Covered

$$1 - \frac{1}{64}$$

Introduction to Series



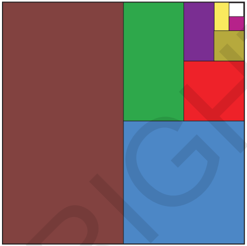
Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

Total Area Covered

$$1 - \frac{1}{128}$$

Introduction to Series



Sum of Areas

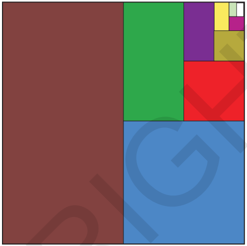
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+ \frac{1}{256}$$

Total Area Covered

$$1 - \frac{1}{256}$$

Introduction to Series



Sum of Areas

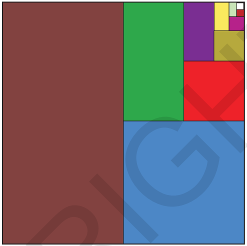
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+\frac{1}{256} + \frac{1}{512}$$

Total Area Covered

$$1 - \frac{1}{512}$$

Introduction to Series



Sum of Areas

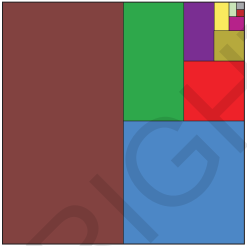
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+\frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$$

Total Area Covered

$$1 - \frac{1}{1024}$$

Introduction to Series



Sum of Areas

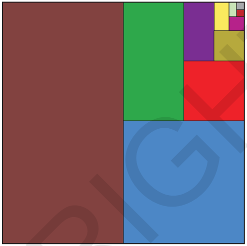
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+ \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots + \frac{1}{2^k}$$

Total Area Covered

$$1 - \frac{1}{2^k}$$

Introduction to Series



Note:

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k} &= \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n} \\ &= \lim_{k\to\infty} 1-\frac{1}{2^k} \\ &= 1\end{aligned}$$

Series

Definition: [Series]

Given a sequence $\{a_n\}$, the *formal* sum

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

is called a *series*. (The series is called *formal* because we have not yet given it a meaning numerically.)

The a_n's are called the *terms* of the series. For each term a_n, the number n is called the *index* of the term.

We denote the series by

$$\sum_{n=1}^{\infty} a_n.$$

Convergent/Divergent Series

Definition: [Convergent Series]

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$, we define the *$k$th partial sum S_k* of the series $\sum_{n=1}^{\infty} a_n$ by

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence of partial sums $\{S_k\}$ converges. In this case, we write

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k$$

Otherwise, we say that the series *diverges* and the sum has no defined value.

Example

Example:

Suppose $a_n = \frac{1}{2^n}$. We know that

$$S_k = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k} \to 1.$$

Hence, $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ converges with

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Why Use Limits?

Question: Why use limits?

Suppose $a_n = (-1)^{n+1}$. Then the formal sum looks like

$$a_1 + a_2 + a_3 + \cdots = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$$

We could parenthesize the formal sum as follows:

$$[1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0.$$

Alternatively, we could parenthesize the formal sum as:

$$1 + [(-1) + 1] + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Our result is ambiguous; the "sum" changes if we change the way we parenthesize the terms!

Why Use Limits?

Observe:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 - 1 = 0 \\ S_3 &= 1 - 1 + 1 = 1 \\ S_4 &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

We get

$$S_k = 1 + (-1) + 1 + \cdots + (-1)^{k+1} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Thus, $\{S_k\} = \{1, 0, 1, 0, 1, 0, \cdots\}$ **diverges**.

Example

Example: Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$

converges or diverges.

Solution: Observe that

$$a_n = \frac{1}{n^2 + n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

so the series becomes

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

We have

$$\begin{aligned}
S_k &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{k} - \frac{1}{k+1}) \\
&= 1 - (\frac{1}{2} - \frac{1}{2}) - (\frac{1}{3} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{4}) - \cdots - (\frac{1}{k} - \frac{1}{k}) - \frac{1}{k+1} \\
&= 1 - 0 - 0 - 0 - \cdots - 0 - \frac{1}{k+1} = 1 - \frac{1}{k+1} \to 1
\end{aligned}$$