Geometric Series

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Geometric Series

Definition: [Geometric Series]

Let $r \in \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

is called a *geometric series* of radius r.

Problem: For which r does the geometric series $\sum_{n=0}^{\infty} r^n$ converge?

To answer this question, we must look at its sequence of partial sums:

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k.$$

Geometric Series

Problem: For which r does the series $\sum_{n=0}^{\infty} r^n$ converge?

Case 1: $r = 1$

$$S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.$$

Since $\{S_k\} = \{k + 1\}$ diverges, the series $\sum_{n=0}^{\infty} 1^n$ diverges.

Geometric Series

Problem: For which r does the series $\sum_{n=0}^{\infty} r^n$ converge?

Case 2: $r = -1$

$$S_k = 1 + (-1) + 1 + \cdots + (-1)^k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Since $\{S_k\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ diverges, $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Geometric Series

Problem: For which r does the series $\sum_{n=0}^{\infty} r^n$ converge?

Case 3: $r \neq 1$

$$\begin{aligned} S_k &= 1 + r + r^2 + \cdots + r^k \\ rS_k &= \phantom{1 + {}} r + r^2 + \cdots + r^k + r^{k+1} \\ \Rightarrow (1-r)S_k &= 1 - r^{k+1} \\ \Rightarrow S_k &= \frac{1 - r^{k+1}}{1-r} \end{aligned}$$

Geometric Series

Problem: For which r does the series $\sum_{n=0}^{\infty} r^n$ converge?

Now

$$| r^{k+1} | \to \begin{cases} 0 & \text{if } | r | < 1, \\ \infty & \text{if } | r | > 1 . \end{cases}$$

But if $S_k = \dfrac{1 - r^{k+1}}{1 - r}$,

$$\lim_{k \to \infty} S_k = \begin{cases} \dfrac{1}{1 - r} & \text{if } | r | < 1, \\ \text{does not exist} & \text{if } | r | \geq 1 . \end{cases}$$

Geometric Series

Theorem: [Geometric Series Test]

A geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$.

Moreover, if $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Example:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$$