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#### **Convergence for Series**

#### **Definition:** [Convergent Series]

Given a sequence  $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ , we define the *kth partial sum*,  $S_k$  of the series

 $\sum_{n=1}^{n}a_{n}$ 

by

$$S_k = a_1 + a_2 + \dots + a_k = \sum_{n=1}^{k} a_n.$$

We say that the series  $\sum_{n=1}^{\infty} a_n$  converges if the sequence of partial sums  $\{S_k\}$  converges. In this case, we write

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k.$$

Otherwise, we say that the series *diverges* and the sum has no defined value.

Problem: Does  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  converge? Observation: Assume that  $\sum_{n=1}^{\infty} a_n$  converges to *L* and that

$$S_k = a_1 + a_2 + \dots + a_k = \sum_{n=1}^k a_n.$$

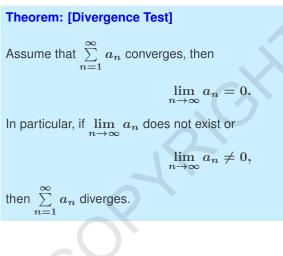
Then

$$S_{k+1} - S_k = \sum_{n=1}^{k+1} a_n - \sum_{n=1}^k a_n = a_{k+1}.$$

It follows that

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} a_{k+1} = \lim_{k \to \infty} S_{k+1} - S_k = \lim_{k \to \infty} S_{k+1} - \lim_{k \to \infty} S_k$$
$$= L - L = 0.$$

If 
$$a_n = rac{n}{n+1}$$
, then  $a_n o 1 
eq 0$ , so  $\sum\limits_{n=1}^{\infty} rac{n}{n+1}$  diverges.



Question: Does the converse hold? That is, if  $\lim_{k \to \infty} a_k = 0$ , does  $\sum_{n=1}^{\infty} a_n$  converge?

**Example:** Consider  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

$$S_{1} = 1 = \frac{2}{2}$$

$$S_{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_{4} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\geq 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4})$$

$$= \frac{4}{2}$$

0

$$S_{8} = S_{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$\geq S_{4}^{\pi} + \frac{\frac{1}{2}}{\frac{1}{5}} + \frac{\frac{1}{8}}{\frac{1}{6}} + \frac{\frac{1}{7}}{\frac{1}{7}} + \frac{1}{8}$$
$$= \frac{5}{2}$$

**Observation:** For each  $j \in \mathbb{N}$ 

$$S_{2^j} \geq \frac{j+2}{2} \to \infty.$$

We have shown that for  $\sum_{n=1}^{\infty} \frac{1}{n}$ , the sequence  $\{S_k\}$  of partial sums is unbounded and therefore divergent. Hence,

even though

$$\lim_{n o \infty} rac{1}{n} = 0.$$

 $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ 

**Definition:** [Harmonic Series]

The divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the Harmonic Series.

**Problem:** How long would it take your computer to add up enough terms in the series

