Arithmetic for Limits of Sequences I

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Arithmetic for Limits of Sequences

Problem:

Suppose that $a_n \to L$ and $b_n \to M$. What can we say about $c_n = 2a_n + b_n$?

We would hope that

$$\begin{aligned} c_n &\to 2\cancelto{L}{a_n} + \cancelto{M}{b_n} \\ &= 2L + M. \end{aligned}$$

Arithmetic for Limits of Sequences

Theorem: [Arithmetic Rules for Limits of Sequences]

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:

i) For any $c \in \mathbb{R}$, if $a_n = c$ for every n, then $c = L$.

ii) For any $c \in \mathbb{R}$, $\lim_{n\to\infty} ca_n = cL$.

iii) $\lim_{n\to\infty} (a_n + b_n) = L + M$.

iv) $\lim_{n\to\infty} a_n b_n = LM$.

v) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

vi) If $a_n \geq 0$ for all n and if $\alpha > 0$, then $\lim_{n\to\infty} a_n^\alpha = L^\alpha$.

vii) For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$.

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Proof of (ii):

For any $c \in \mathbb{R}$, if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} ca_n = cL$.

Case 1: $c = 0$

Let $c_n = c \cdot a_n$. Then

$$c_n = 0 \cdot a_n = 0$$

for all $n \in \mathbb{N}$. So

$$c_n \to 0 = c \cdot L.$$

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Proof of (ii) continued:

For any $c \in \mathbb{R}$, if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} ca_n = cL$.

Case 2: $c \neq 0$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that if $n \geq N$, then

$$| a_n - L | < \frac{\epsilon}{| c |}.$$

If $n \geq N$, then

$$\begin{aligned} | ca_n - cL | &= | c || a_n - L | \\ &< \cancel{| c |}\frac{\epsilon}{\cancel{| c |}} \\ &= \epsilon. \end{aligned}$$

Arithmetic for Limits of Sequences

Key Observation:

$$\text{Error of } [c\cdot \text{ Quantity}] \approx c\cdot \text{ Error}$$

Arithmetic of Limits

Proof of (iii): Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$.

Let $\epsilon > 0$. Choose $N_1 \in \mathbb{N}$ so that if $n \geq N_1$,

$$| a_n - L | < \frac{\epsilon}{2}.$$

Choose $N_2 \in \mathbb{N}$ so that if $n \geq N_2$,

$$| b_n - M | < \frac{\epsilon}{2}.$$

Let $N_0 = \max\{N_1, N_2\}$. If $n \geq N_0$

$$\begin{aligned} | (a_n + b_n) - (L + M) | &= | (a_n - L) + (b_n - M) | \\ &\leq | a_n - L | + | b_n - M | \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Arithmetic for Limits of Sequences

Key Observation:

Error of a Sum $\approx$ Sum of the Errors

Limits of the Form $\frac{0}{0}$

Remark: Rule v) says that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$$

if $M \neq 0$.

However, if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ may or may not exist.

Examples:

- $a_n = \frac{1}{n}, b_n = \frac{1}{n} \Rightarrow \frac{a_n}{b_n} = 1 \to 1.$
- $a_n = \frac{1}{n}, b_n = \frac{1}{n^2} \Rightarrow \frac{a_n}{b_n} = \frac{\frac{1}{n}}{\frac{1}{n^2}} = n \to \infty.$
- $a_n = \frac{1}{n^2}, b_n = \frac{1}{n} \Rightarrow \frac{a_n}{b_n} = \frac{\frac{1}{n^2}}{\frac{1}{n}} = \frac{1}{n} \to 0.$

Limits of the Form $\frac{0}{0}$

Observation: Assume that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals L, and $\lim_{n\to\infty} b_n = 0$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} b_n = L \cdot 0 = 0.$$

Theorem:

Assume that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and that $\lim_{n\to\infty} b_n = 0$. Then

$$\lim_{n\to\infty} a_n = 0.$$