Arithmetic for Limits of Sequences

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Limits of the Form $\frac{p(n)}{q(n)}$

Let

$$a_n = \frac{b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \cdots + b_j n^j}{c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k}.$$

By factoring out n^j from the numerator and n^k from the denominator, and then rewriting the sequence as

$$a_n = \frac{n^j}{n^k} \left[\frac{\frac{b_0}{n^j} + \frac{b_1}{n^{j-1}} + \frac{b_2}{n^{j-2}} + \frac{b_3}{n^{j-3}} + \cdots + b_j}{\frac{c_0}{n^k} + \frac{c_1}{n^{k-1}} + \frac{c_2}{n^{k-2}} + \cdots + c_k} \right],$$

we can conclude that

$$\lim_{n \to \infty} a_n = \begin{cases} \frac{b_j}{c_k} & \text{if } j = k \\ 0 & \text{if } j < k \\ \infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} > 0 \\ -\infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} < 0. \end{cases}$$

Examples

Example 4: Evaluate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n.$$

Note:

i) $\lim_{n\to\infty} \sqrt{n^2+n} - \lim_{n\to\infty} n = \infty - \infty$ has no meaning.

ii) $a_{100} = 0.498756211,$
$a_{10^6} = 0.499999875 \Rightarrow \lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$?

We have

$$\begin{aligned}
\sqrt{n^2+n} - n &= (\sqrt{n^2+n} - n) \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \frac{(n^2+n) - n^2}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \\
&= \frac{n}{n} \cdot \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}
\end{aligned}$$

Finding Limits for Recursive Sequences

Example 5:

Let $a_1 = 1$ and

$$a_{n+1} = \frac{1}{1 + a_n}.$$

Note: It appears that $\{a_n\}$ converges.

Problem: What is

$$\lim_{n\to\infty} a_n?$$

n	a_n	Decimal
1	1	1.0000000
2	$\frac{1}{2}$	.5000000
3	$\frac{2}{3}$	.6666666
4	$\frac{3}{5}$	.6000000
5	$\frac{5}{8}$	.6250000
6	$\frac{8}{13}$	.6153846
7	$\frac{13}{21}$	.6190476
8	$\frac{21}{34}$	.6176470
9	$\frac{34}{55}$	.6181818
10	$\frac{55}{89}$	.6179775
11	$\frac{89}{144}$	.6180555
12	$\frac{144}{233}$	.6180257
13	$\frac{233}{377}$	.6180371
14	$\frac{377}{610}$	.6180327
15	$\frac{610}{987}$	.6180344
16	$\frac{987}{1597}$	.6180338

Finding Limits for Recursive Sequences

Solution: Assume that

$$\lim_{n\to\infty} a_n = L.$$

We have

$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \frac{1}{1+a_n} \\ &= \frac{1}{1+\lim_{n\to\infty} a_n} \\ &= \frac{1}{1+L}. \end{aligned}$$

Hence,

$$L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2}.$$

Since each $a_n > 0$,

$$L = \frac{-1+\sqrt{5}}{2} = 0.618033989\ldots.$$