Applications of computable model theory to computable analysis

Alexander Melnikov

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Waterloo, May 2013
Idea (v.d.Waerden, F. and S., Mal’cev, Rabin)

Computable mathematics should study computable mathematical objects up to computable isomorphisms.
The beginning of the story

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Rabin and Mal’cev did not do much of computable analysis.
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These notions have been developed almost independently from computable model theory and computable algebra.
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Based on this idea and similar ideas, we aim to develop a new approach to computable analystis.
In this talk, a **space** is a separable metric structure

\[(M, d, F_1, \ldots, F_n, \ldots),\]

where \(d\) is a metric, and \(F_1, \ldots, F_n, \ldots\) are distinguished points or operations on \(M\).
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- Banach algebra \((d, +, 0, 1, \times, (r \cdot)_{r \in \mathbb{Q}})\);
- Whatever (lattice operations, another metric, inner product, or your favorite collection of operators.)
A dense computable sequence \((q_i)_{i\in\mathbb{N}}\) in a space \(\mathcal{M}\) is a **computable structure** on \(\mathcal{M}\) if:

1. \(d(q_i, q_j)\) is a computable real uniformly in \(i\) and \(j\), and
2. the distinguished points and operations are uniformly computable in this structure.
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*Our definition depends on the choice of signature.*
Example (The reals)

Let \((q_i)_{i \in \mathbb{N}}\) be an effective listing of rationals and \(\gamma\) a real. For any \(\gamma\), the collection \((q_i + \gamma)_{i \in \mathbb{N}}\) is a computable structure on \((\mathbb{R}, d)\), where \(d\) is the Euclidean metic.
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- Now consider \((\mathbb{R}, d, 0, +)\). The collection \((q_i + \gamma)_{i \in \mathbb{N}}\) is a computable structure on \((\mathbb{R}, d, 0, +)\) only if \(\gamma\) is a computable real.
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- Polynomials with rational coefficients make \((C[0, 1], \text{sup})\) a computable Banach algebra.
Elementary examples

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Example (Continuous functions)

- Polynomials with rational coefficients make \((C[0, 1], \text{sup})\) a computable Banach algebra.

- A non-computable “shift” keeps \((C[0, 1], \text{sup})\) a computable metric space, but not a computable Banach space.
Use your intuition to define what is a computable map between computable spaces.
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**Definition (M., Pour-El and Richards for Banach spaces.)**

A space $\mathcal{M}$ is *computably categorical* if every two computable structures on $\mathcal{M}$ are equivalent up to a computable map between completions of these computable structures.
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Please draw a picture.

**Example**

The structures $(q_i)_{i \in \mathbb{N}}$ and $(q_i + \gamma)_{i \in \mathbb{N}}$ on the space $(\mathbb{R}, d)$ agree up to the isometry $x \rightarrow x + \gamma$. 
The problems

Problem
Which spaces are computably categorical?

Problem
If a space is not computably categorical, how many computable structures may it have (up to computable automorphisms)?
Known facts:

- Every separable Hilbert space \((H, d, +, (r \cdot r))\) is computably categorical. [M., 2012] So is the associated metric space \((H, d)\).

- The Banach space \(l_1\) is not computably categorical. [M., 2012] So is the associated metric space.

- There exists a Polish space having exactly two computable structures (up to a computable surj. self-isometry). [Khoussainov and M., unpublished]

- Cantor space \((C, d)\) with the usual ultra-metric is computably categorical. The Urysohn space is computably categorical. [M., 2012]

- The space \((C[0,1], \sup)\) with the usual pointwise supremum metric is not computably categorical.
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**Theorem (M., Ng)**

The space $(C[0,1], sup)$ has infinitely many structures pairwise non-equivalent up to automorphisms.
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*(The first main idea)*

There is a computable structure on $(C[0,1], \text{sup})$ such that the pointwise $+$ is not computable in this structure (M., 2012). We construct a structure with this property which is additionally “$\Delta^0_2$-isometric” (in some restricted sense) to the standard computable structure.
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**(The second main idea)**

We take Goncharov’s sufficient condition for a countable structure to have comp. dim. $\omega$, and then merge Goncharov’s strategy with a certain analytic requirement.
A closer look at \((C[0,1], \text{sup})\)

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Proof idea.

Make the pointwise multiplication \(\times\) \text{ non-computable in your structure.}
What if we also add $\times$ into the signature? (Make it a Banach algebra!)

Theorem (M., Ng)

The Banach algebra $(C[0,1], \sup, +, (r \cdot))$ with $r \in \mathbb{Q}$, $\times$ is not computably categorical.

Fact (M., Ng)

There is a distinguished point which, when also added into the signature, makes the Banach algebra $C[0,1]$ computably categorical.

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We can define polynomials with rational coefficients using the function $f(x) = x$.
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Intermediate conclusions

If a computable structure on \( C[0, 1] \) does not compute polynomials with rational coefficients, we get a pathology.

Our studies are related to intrinsic computability of operations on metric spaces. (The question of when an operation is computable in every computable structure on the space.)
Recall:

**Idea**

The complexity of a computable object is reflected in the complexity of isomorphisms/automorphisms of the object.

Recall also we had an application of $\Delta^0_2$ isometries to the number of computable structures on $\mathbb{C}[0, 1]$.

(Ash, Goncharov, Knight, and many others)

We should study computable structures categorical relative to an oracle.
In computable model theory, $\Delta^0_n$-isomorphisms have been studied by various authors:

- Well-orderings (Ash)
- Linear orders (McCoy, Downey)
- Boolean algebras (Knight, McCoy, Harris)
- Fields (Miller, Kudinov)
- Abelian groups (Barker, Morozov, Harizanov, Calvert, Downey, M.)

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Every compact computable metric space is $\Delta^0_3$-categorical.

It means that we can build an isometry with a help of $\emptyset''$. In fact, $\emptyset''$ can be improved to low relative to $\emptyset'$, but provably cannot be improved to $\emptyset'$. Nies and I also showed that every compact c.m.s. has a c.e. Scott family consisting of $\Pi^2_2$ computable infinitary formulas. (A compact c.m.s. can be described by a single computable $\Sigma^3_3$ infinitary Scott sentence.) Although we both feel this fact is closely related to (relative) $\Delta^0_3$-categoricity, we don't know why and how exactly.
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Lots of problems

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What can be said about relative computable ($\Delta^0_n$-) categoricity and intrinsic computability of relations and operations? How is it related to infinitary logic? What is the right language to use?

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SPASIBO!