What is Categorical Type Theory?

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Introduction

Motivation: To understand Martin-Löf type theory.

Conceptual mathematics $\rightarrow$ category theory.

Two questions:

- Is type theory soluble in categorical logic?
- Is category theory soluble in type theory?

I will not discuss the second question here.

Overview

Aspects of categorical logic:

- Cartesian closed categories
- Essentially algebraic theories
- Locally cartesian closed categories
- Tribes

Homotopical logic:

- Weak factorization systems
- Homotopical algebra
- Pre-typoi
- Typoi
- Univalent typoi

Aspects of categorical logic

The basic principles of categorical logic was expressed in Lawvere's paper *Adjointness in Foundation* (1969). I will use these principles implicitly.

- Algebraic theories
- Cartesian closed categories
- Essentially algebraic theories
- Locally cartesian closed categories
- Tribes
- Π-tribes

Monoids and groups in categories(1)

Eckmann and Hilton: groups and cogroups in the category of pointed topological spaces ($\sim$ 1962).

Diagrammatic language:

A *monoid object* in a category $\mathcal{C}$ is an object $A \in \mathcal{C}$ equipped with two operations $\mu : A \times A \to A$ and $e : \top \to A$ such that the following diagrams commute,

Associativity:

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mu \times A"] \arrow[d, "A \times \mu"'] & A \times A \arrow[d, "\mu"] \\ A \times A \arrow[r, "\mu"] & A. \end{tikzcd}$$

Left and right units,

$$\begin{tikzcd} A \arrow[r, "e \times A"] \arrow[dr, equal, "1_A"'] & A \times A \arrow[d, "\mu"] \\ & A, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A \times e"] \arrow[dr, equal, "1_A"'] & A \times A \arrow[d, "\mu"] \\ & A. \end{tikzcd}$$

Monoids and groups in categories(2)

A *group object* in $\mathcal{C}$ is a monoid (G, μ, e) equipped with an operation of *inversion* $\theta : G \to G$ such that the following diagrams commute,

$$\begin{tikzcd} G \times G \arrow[rr, "\theta \times G"] & & G \times G \arrow[d, "\mu"] \\ G \arrow[u, "(1_G, 1_G)"] \arrow[r] & \top \arrow[r, "e"] & G \end{tikzcd}$$

$$\begin{tikzcd} G \times G \arrow[rr, "G \times \theta"] & & G \times G \arrow[d, "\mu"] \\ G \arrow[u, "(1_G, 1_G)"] \arrow[r] & \top \arrow[r, "e"] & G \end{tikzcd}$$

Cartesian product

Recall that the *cartesian product* $X \times Y$ of two objects X and Y in a category $\mathcal{C}$ is an object $X \times Y$ equipped with a pair of maps

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

having the following universal property: for any object $C \in \mathcal{C}$ and any a pair of maps

$$X \xleftarrow{f} C \xrightarrow{g} Y,$$

there is a unique map $h = \langle u, v \rangle : C \to X \times Y$ such that $p_1 h = u$ and $p_2 h = v$.

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

Cartesian category

A category $\mathcal{C}$ is said to be *cartesian* if it has finite cartesian products

Recall that an object $\top$ is said to be *terminal* if for every object $A \in \mathcal{C}$, there is a unique map

$$! : A \to \top.$$

We may say that a map $a : \top \to A$ is a *point* of A, or a *term* of type A and write $a : A$.

A functor between cartesian categories $F : \mathcal{C} \to \mathcal{D}$ is said to be *cartesian* if it preserves (finite) cartesian products.

Algebraic theories (1)

Lawvere (1963):

- An algebraic theory $\mathbb{T}$ generates cartesian category $\mathcal{C}(\mathbb{T})$;
- the objects of $\mathcal{C}(\mathbb{T})$ are finite products of sorts $U \times V \times W$, ...;
- the maps in $\mathcal{C}(\mathbb{T})$ are the operations of $\mathbb{T}$;
- two maps $f, g : X \to Y$ in $\mathcal{C}(\mathbb{T})$ are equal if they are *provably* equal operations in $\mathbb{T}$;
- a model of $\mathbb{T}$ is a cartesian functor $M : \mathcal{C}(\mathbb{T}) \to \mathbf{Set}$;
- A morphism of models $F \to G$ is a natural transformation.

Completeness theorem: two maps $f, g : X \to Y$ in $\mathcal{C}(\mathbb{T})$ are equal if and only if $M(f) = M(g)$ for every model $M : \mathcal{C}(\mathbb{T}) \to \mathbf{Set}$.

Follows from Yoneda lemma!

Algebraic theories (2)

Example: The cartesian theory of monoids *Mon* is the cartesian category generated by one object $A \in Mon$ and two operations $\mu : A \times A \to A$ and $e : \top \to A$ (the multiplication and unit) satisfying:

Associativity:

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mu \times A"] \arrow[d, "A \times \mu"'] & A \times A \arrow[d, "\mu"] \\ A \times A \arrow[r, "\mu"] & A \end{tikzcd}$$

Left and right units,

$$\begin{tikzcd} A \arrow[r, "e \times A"] \arrow[dr, equal, "1_A"'] & A \times A \arrow[d, "\mu"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A \times e"] \arrow[dr, equal, "1_A"'] & A \times A \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

Cartesian closed categories (1)

If A and B are two objects of a cartesian category $\mathcal{C}$, we shall say that an object $[A, B]$ equipped with a map $\epsilon : [A, B] \times A \to B$ is the *exponential* of B by A if for every object $C \in \mathcal{C}$ and every map $f : C \times A \to B$ there exists a unique map $\lceil f \rceil : C \to [A, B]$ such that $\epsilon(\lceil f \rceil \times A) = f$.

$$
\begin{tikzcd}
C \times A \arrow[r, "\lceil f \rceil \times A"] \arrow[dr, "f"'] & {[A, B] \times A} \arrow[d, "ev"] \\
 & B
\end{tikzcd}
$$

We shall write $\lambda^A f = \lceil f \rceil$. This defines a natural bijection between

the maps	$C \times A \to B$
and the maps	$C \to [A, B]$.

Cartesian closed categories (2)

A cartesian category $\mathcal{C}$ is said to be *closed* if the object $[A, B]$ exists for every pair of objects $A, B \in \mathcal{C}$.

Examples of cartesian closed categories

- the category of sets **Set**
- the category of (small) catégories **Cat** (Lawvere)
- the category of groupoids **Grpd**
- Every category $[\mathbb{C}, \mathbf{Set}]$
- The category of simplicial sets $[\Delta^{op}, \mathbf{Set}]$

Essentially algebraic theories (1)

Ehresmann: Theory of sketches ($\sim$1968),

Gabriel and Ulmer: Locally presentable categories ($\sim$ 1971)

Slice categories

Recall that the *slice category* $\mathcal{C}/A$ has for objects the pairs (X, p), where p is a map $X \to A$ in $\mathcal{C}$. The map $p : X \to A$ is called the *structure map* of (X, p).

A morphism $(X, p) \to (Y, q)$ in $\mathcal{C}/A$ is a map $u : X \to Y$ in $\mathcal{C}$ such that $qu = p$,

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & A. & \end{tikzcd}$$

Push-forward

To every map $f : A \to B$ in a category $\mathcal{C}$ we can associate a *push-forward* functor

$$f_! : \mathcal{C}/A \to \mathcal{C}/B$$

by putting $f_!(X, p) = (X, fp)$ for every map $p : X \to A$,

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "p"'] & X \arrow[d, "fp"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Fiber products

Recall that the *fiber product* of two maps $X \to A$ and $Y \to A$ in a category $\mathcal{C}$ is their cartesian product $X \times_A Y$ as objects of the category $\mathcal{C}/A$.

$$\begin{tikzcd} X \times_A Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "q"] \\ X \arrow[r, "p"] & A. \end{tikzcd}$$

The square is also called a *pullback square*.

Base changes

In a category with finite limits $\mathcal{C}$ the push-forward functor $f_! : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint

$$f^\star : \mathcal{C}/B \to \mathcal{C}/A$$

for any map $f : A \to B$. The functor $f^\star$ takes a map $p : X \to B$ to the map $p_1 : A \times_B X \to A$ in a pullback square

$$\begin{tikzcd}
A \times_B X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

The map p_1 is said to be the *base change* of the map $p : X \to B$ *along* the map $f : A \to B$.

Essentially algebraic theories (1)

An *essentially algebraic theory* is defined to be a category with finite limits.

A *morphism* of essentially algebraic theories $F : \mathcal{C} \to \mathcal{D}$ is defined to be a functor which preserves finite limits.

A *model* of an essentially algebraic theory $\mathcal{C}$ is defined to be a morphism $M : \mathcal{C} \to \mathbf{Set}$.

For example, the notion of *category* is essentially algebraic, but not algebraic.

Duality

Let $\mathcal{C}$ be a category with finite limits.

For every $A \in \mathcal{C}$ the functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ is a model of $\mathcal{C}$, since it preserves finite limits.

In fact, the functor $A \mapsto \mathcal{C}(A, -)$ is a contravariant equivalence between the category $\mathcal{C}$ and the category of finitely presentable models of $\mathcal{C}$.

For example, the finite limit theory of categories is the opposite of the category of finitely presentable categories.

Locally cartesian closed categories(1)

A category with finite limits $\mathcal{C}$ is said to be *locally cartesian closed* if the category $\mathcal{C}/A$ is cartesian closed for every object $A \in \mathcal{C}$.

A category with finite limits $\mathcal{C}$ is locally cartesian closed if and only if the base change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint

$$f_\star : \mathcal{C}/A \to \mathcal{C}/B$$

for every map $f : A \to B$ in $\mathcal{C}$. The functor $f_\star$ is a *internal product along* $f : A \to B$,

$$f_\star = \Pi_f : \mathcal{C}/A \to \mathcal{C}/B,$$

Locally cartesian closed categories(2)

Examples of locally cartesian closed categories

- the category of sets **Set**
- Every category $[\mathbb{C}, \mathbf{Set}]$
- The category of simplicial sets $[\Delta^{op}, \mathbf{Set}]$
- A Grothendieck topos

The category **Cat** is *not* locally cartesian closed.

The category **Grpd** is *not* locally cartesian closed.

Tribes(1)

A class of maps $\mathcal{F}$ in a category $\mathcal{C}$ is said to be *closed under base changes* if the base change $A \times_B X \to A$ of every map $X \to B$ in $\mathcal{F}$ along any map $f : A \to B$ in $\mathcal{C}$ exists and belongs to $\mathcal{F}$,

Definition

A *tribe structure* on a category with terminal object $\mathcal{C}$ is a class of maps $\mathcal{F} \subseteq \mathcal{C}$ which satisfies the following conditions:

- every isomorphism belongs to $\mathcal{F}$;
- $\mathcal{F}$ is closed under composition and base changes;
- the map $X \to \top$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

A map in $\mathcal{F}$ is a *family* or a *fibration* of the tribe.

A *tribe* is a category $\mathcal{C}$ equipped tribe structure $\mathcal{F}$.

Tribes(2)

The *fiber* of a fibration $p : X \to A$ at a point $a : \top \to A$ is the object $X(a)$ defined by the pullback square

$$
\begin{tikzcd}
X(a) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\top \arrow[r, "a"] & A.
\end{tikzcd}
$$

The fibration $p : X \to A$ can be regarded as an *internal family of objects* $(X(a) : a \in A)$ parametrized by the codomain of p.

We denote by $\mathcal{C}(A)$ the full subcategory of $\mathcal{C}/A$ whose objects are the fibrations $X \to A$. An object of $\mathcal{C}(A)$ is an *internal family* of objects of A.

Tribes (3)

If $u : A \to B$ is a map in $\mathcal{C}$, then the base-change functor

$$u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$$

is an operation of *substitution of parameters* for internal families, since we have

$$u^\star(Y)(a) = Y(u(a))$$

for every every fibration $Y \to B$ and every $a \in A$.

Tribes (4)

To every fibration $f : A \to B$ in tribe $\mathcal{C}$ we can associate a *push-forward* functor $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$ by putting $f_!(X, p) = (X, fp)$,

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "p"'] & X \arrow[d, "fp"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

The functor $f_!$ is a *summation along* f,

$$f_! = \Sigma_f : \mathcal{C}(A) \to \mathcal{C}(B).$$

Formally,

$$f_!(X)(b) = \sum_{f(a)=b} X(a)$$

for every fibration $X \to A$ and every $b \in B$.

Tribes (5)

If $\mathcal{C}$ is a tribe, then the category $\mathcal{C}(A)$ has the structure of a tribe for every object $A \in \mathcal{C}$.

By definition, a morphism $f : (X, p) \to (Y, q)$ in $\mathcal{C}(A)$ is a fibration if the map $f : X \to Y$ is a fibration in $\mathcal{C}$.

Tribes (6)

Definition

A *morphism of tribes* $F : \mathcal{C} \to \mathcal{D}$ is a functor which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of tribes for any map $u : A \to B$ in a tribe $\mathcal{C}$.

Tribes are essentially algebraic theories

A *set model* of a tribe $\mathcal{C}$ is a functor $F : \mathcal{C} \to Set$ preserving the pullback squares defined by base-changes of fibrations.

For example, the theory of categories takes the form of a tribe $\mathbb{T}(Cat)$ if the map

$$(s, t) : A \to O \times O$$

is declared to be a fibration.

The objects and morphisms in $\mathbb{T}(Cat)$ can be described explicitly.

The equality relation between two maps in $\mathbb{T}(Cat)$ is decidable.

Π-tribes (1)

Definition

We say that a tribe $\mathcal{C}$ is a Π-*tribe* if the product of a fibration $X \to A$ along any fibration $f : A \to B$ exists and the structure map $\Pi_f(X) \to B$ is a fibration.

$$\begin{tikzcd} X \arrow[d, "\in\mathcal{F}"] & \Pi_f(X) \arrow[d, "\in\mathcal{F}"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

If $\mathcal{C}$ is a Π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Π-tribes (2)

If $f : A \to B$ is a fibration in a Π-tribe $\mathcal{C}$, then the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ has a right adjoint $f_\star$, The functor $f_\star$ is a *product along* f,

$$f_\star = \Pi_f : \mathcal{C}(A) \to \mathcal{C}(B).$$

Formally,

$$f_*(X)(b) = \prod_{f(a)=b} X(a)$$

for every fibration $X \to A$ and every $b \in B$.

A Π-tribe is cartesian closed.

Π-tribes (3)

Example of Π-tribes.

A locally cartesian closed category is a Π-tribe in which every map is a fibration.

Definition

A functor between groupoids $F : \mathbb{A} \to \mathbb{B}$ is an *iso-fibration* if for every object $A \in \mathbb{A}$ and every morphism $g \in \mathbb{B}$ with domain $F(A)$ there exists a morphism $f \in \mathbb{A}$ with domain A such that $F(f) = g$.

The category of small groupoids **Grpd** becomes a Π-tribe if the fibrations are taken to be the iso-fibrations.

Π-tribes (4)

Definition

A *morphism of* Π*-tribes* $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the internal product $\Pi_f(X)$.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of Π-tribes for any map $u : A \to B$ in a Π-tribe $\mathcal{C}$.

Homotopical logic

- Weak factorization systems
- Quillen model categories
- Pre-typoi
- Typoi
- Univalent typoi

Weak factorisation systems(1)

Definition

A map $u : A \to B$ in a category $\mathcal{C}$ is said to have the *left lifting property* with respect to a map $f : X \to Y$, and f is to have the *right lifting property with respect to u*, if every commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

has a diagonal filler $d : B \to X$, $du = a$ and $fd = b$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[ur, "d"] & Y \end{tikzcd}$$

We shall denote this relation by $u \pitchfork f$.

Weak factorisation systems(2)

For any class of maps $\mathcal{S} \subseteq \mathcal{C}$, let us put

$$\mathcal{S}^{\pitchfork} = \{f \in \mathcal{C} : \forall u \in \mathcal{S} \quad u \pitchfork f\}$$

$$^{\pitchfork}\mathcal{S} = \{u \in \mathcal{C} : \forall f \in \mathcal{S} \quad u \pitchfork f\}$$

Definition

A pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in a category $\mathcal{C}$ is said to be a *weak factorization system* if the following two conditions are satisfied

- $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$
- Every map $f : A \to B$ in $\mathcal{C}$ admits a factorization $f = pu : A \to E \to B$ with $u \in \mathcal{L}$ and $p \in \mathcal{R}$.

Homotopical algebra(1)

Recall that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ is said to have the *3-for-2 property* if whenever two sides of a commutative triangle

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "uv"'] & B \arrow[d, "v"] \\ & C \end{tikzcd}$$

belongs to $\mathcal{W}$, then so is the third (3 apples for the price of two!).

Homotopical algebra(2)

Quillen (1967)

Definition

Recall that a *Quillen model structure* on a category $\mathcal{E}$ consists on three class of maps $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ respectively called the *cofibrations*, the *weak equivalences* and the *fibrations*, such that the following conditions are satisfied:

- $\mathcal{W}$ has the 3-for-2 property;
- the pair $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ is a weak factorisation system;
- the pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a weak factorisation system.

A *model category* is a category $\mathcal{E}$ equipped with a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

Homotopical algebra and type theory(1)

Theorem (Awodey-Warren):

Martin-Löf type theory can be interpreted in a model category:

- types are interpreted as fibrant objects;
- display maps are interpreted as fibrations;
- the identity type $Id_A \to A \times A$ is a path object for A;
- the reflexivity term $r : A \to Id_A$ is an acyclic cofibration.

Homotopical algebra and type theory(2)

Let $\mathcal{C}(\mathbb{T})$ be the syntactic category of Martin-Löf type theory.

Let $\mathcal{F}$ be the class of display maps in $\mathcal{C}(\mathbb{T})$.

Theorem (Gambino-Garner):

Every map $f : A \to B$ in $\mathcal{C}(\mathbb{T})$ admits a factorization
$f = pu : A \to E \to B$ with $u \in {}^{\pitchfork}\mathcal{F}$ and $p \in \mathcal{F}$.

Pre-typoi(1)*

We say that a map in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is *anodyne* if it belongs to the class ${}^{\pitchfork}\mathcal{F}$.

Definition

We say that a tribe $\mathcal{C}$ is a *pre-typos** if the following two conditions are satisfied

- the base change of an anodyne map along a fibration is anodyne;
- every map $f : A \to B$ admits a factorization $f = pu : A \to E \to B$ with u an anodyne map and p a fibration.

$(\star)$ Any idea for a better name?

Pre-typoi(2)

A *path object* for an object X in a pre-typos $\mathcal{C}$ is a factorisation

$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal $X \to X \times X$ as an anodyne map $\sigma : X \to PX$ followed by a fibration $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$.

$$\begin{tikzcd}
 & & X \\
X \arrow[r, "\sigma"] \arrow[urr, bend left, "1_X"] \arrow[drr, bend right, "1_X"'] & PX \arrow[ur, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
 & & X
\end{tikzcd}$$

Pre-typoi(3)

A *homotopy* between two maps $f, g : X \to Y$ is a map $H : X \to PY$ such that $\partial_0 H = f$ and $\partial_1 H = g$,

$$\begin{tikzcd}
 & & Y \\
X \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & PY \arrow[ur, "\partial_0"] \arrow[dr, "\partial_1"'] & \\
 & & Y
\end{tikzcd}$$

We shall write $H : f \sim g$ or $f \sim g$.

Pre-typoi(4)

The homotopy relation $f \sim g$ is a congruence on the arrows of the category $\mathcal{C}$.

This defines a category $Ho(\mathcal{C}) = \mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is a *homotopy equivalence* if it is invertible in $Ho(\mathcal{C})$.

An object $X \in \mathcal{C}$ is *contractible* if the map $X \to \top$ is a homotopy equivalence.

Pre-typoi(5)

Definition

A *morphism of pre-typoi* $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of pre-typoi for any map $u : A \to B$ of a pre-typos $\mathcal{C}$.

Typoi(1)*

Definition

A pre-typos $\mathcal{C}$ is a *typos** if it is a Π-tribe and the product functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ preserves the homotopy relation for every fibration $f : A \to B$.

If $\mathcal{C}$ is a typos, then so is the tribe $\mathcal{C}(A)$ for any object $A \in \mathcal{C}$.

$(\star)$ Named after a joke by Steve Awodey. Do you know a better name?

Typoi(2)

Theorem (Hoffman and Streicher)

The category of groupoids **Grpd** has the structure of a typos in which the fibrations are the isofibrations.

Typoi(3)

Definition

A *morphism of typoi* $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation;
- the internal products $\Pi_f(X)$.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of typoi for any map $u : A \to B$ in a typos $\mathcal{C}$.

Typoi(4)

If $u : A \to B$ is a map in a typos $\mathcal{C}$, then the functor

$$Ho(u^\star) : Ho(\mathcal{C}(B)) \to Ho(\mathcal{C}(A))$$

has a both a left adjoint and a right adjoint.

The functor

$$A \mapsto Ho(\mathcal{C}(A))$$

is a hyper-doctrine in the sense of Lawvere!

Typoi(5)

A typos $\mathcal{C}$ may contain a sub-typos of *small fibrations*.

A small fibration $p : U' \to U$ is *universal* if for every small fibration $f : X \to B$ there exists a (homotopy) cartesian square:

$$\begin{tikzcd} X \arrow[r, "\chi'"] \arrow[d, "f"'] & U' \arrow[d, "p"] \\ B \arrow[r, "\chi"] & U. \end{tikzcd}$$

Martin-Löf axiom: *There exists a universal fibration U.*

Univalent typoi

The pair (χ, χ') is *classifying* the fibration $f : X \to B$.

A fibration $p : U' \to U$ is *univalent* if the pair (χ, χ') classifying a fibration is *homotopy unique*.

Voevodsky axiom: *The universal fibration $U' \to U$ is univalent.*

Theorem (Voevodsky)

The category of Kan complexes **Kan** has the structure of a univalent typos in which the fibrations are the Kan fibrations.

Conclusions

Homotopy type theory depends on the notion of weak factorization system, at least implicitly.

The notion of weak factorization system first appeared in homotopical algebra; it arose from a *praxis*, not from adjointness principles.

Homotopy type theory is soluble in category theory, but not in a logical framework exclusively based on adjointness principles.

Thanks !