MATROIDS REPRESENTABLE OVER FIELDS WITH A COMMON SUBFIELD

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Abstract. A matroid is GF($q$)-regular if it is representable over all proper superfields of the field GF($q$). We show that, for highly connected matroids having a large projective geometry over GF($q$) as a minor, the property of GF($q$)-regularity is equivalent to representability over both GF($q^2$) and GF($q^t$) for some odd integer $t \geq 3$. We do this by means of an exact structural description of all such matroids.

1. Introduction

For a field $\mathbb{F}_0$, we say a matroid $M$ is $\mathbb{F}_0$-regular if $M$ is representable over every field $\mathbb{F}$ having $\mathbb{F}_0$ as a proper subfield.

Let $n \geq 2$ be an integer, $q$ be a prime power, and $N$ be a PG($n-1, q$)-restriction of a matroid $M \cong$ PG($n-1, q^2$). Let $L_0$ be a line of $N$ and $x \in cl_M(L_0) - L_0$. We denote by $\hat{\text{PG}}(n-2, q)$ any matroid isomorphic to $\text{si}((M/x)|E(N))$. If $n \geq 3$ and $f \in E(N) - L_0$, then we denote by $\text{PG}(n-1, q)$ any matroid isomorphic to $M|(E(N)\cup\text{cl}_M(\{x, f\}))$. (We will show later that these matroids are uniquely determined up to isomorphism.) A matroid $M$ is round if $E(M)$ is not the union of two hyperplanes, or equivalently if $M$ is infinitely vertically connected. Our main theorem is the following:

Theorem 1.1. Let $q$ be a prime power and $M$ be a round rank-$r$ matroid with a PG($12q^{12} + 19, q$)-minor. The following are equivalent:

1. $M$ is GF($q$)-regular;
2. $M$ is representable over GF($q^2$) and GF($q^t$) for some odd integer $t \geq 3$; and
3. $\text{si}(M)$ is a restriction of either $\hat{\text{PG}}(r-1, q)$ or $\text{PG}(r-1, q)$.

This exactly characterises all GF($q$)-regular matroids that are sufficiently ‘rich’ and highly connected; the equivalence of [1] and [2] is strongly reminiscent of Tutte’s characterisation of regular matroids of the usual sort, and motivates our use of the word. This equivalence

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may hold for all matroids (this has essentially been conjectured for $q = 2$ in [9, Conjecture 6.8]), but the characterisation in [3] requires some extra hypotheses, and we briefly discuss the ones we chose.

As one could otherwise construct counterexamples by taking 2-sums and 3-sums, some connectivity assumption is needed. However, the hypothesis of roundness is probably overkill. The theorem likely holds for vertically 4-connected matroids, and many of our techniques apply in this more general setting. Proving a ‘vertically 4-connected’ version of the theorem would require analysis of how the structure in [3] propagates over 4-separations.

The hypothesis of having some sort of underlying ‘richness’, here a large projective geometry minor, is also necessary; the structure in [3] does not describe all vertically 4-connected GF($q$)-regular matroids. Indeed, Gerards [6] defined a class of signed-graphic matroids representable over every field with at least three elements; this class contains counterexamples to our theorem of arbitrarily high branch-width. However, Gerards’ counterexamples are nearly planar; it is possible that a very similar structure to that in [3] holds for all vertically 4-connected matroids with a large enough clique minor. Round GF($q^2$)-representable matroids of huge rank have a large clique minor [4], so in the round setting it is possible that our hypothesis of a large projective geometry minor could be replaced with a ‘large rank’ hypothesis with few other changes to the theorem statement.

Though the material in this paper is self-contained, sections 6 and 7 make essential use of the theory of tangles and some currently unpublished techniques due to Geelen, Gerards and Whittle [5].

2. Preliminaries

We largely follow the notation of Oxley [8]. We also write $\epsilon(M)$ for $|\text{si}(M)|$. For a positive integer $n$, we denote the set $\{1, \ldots, n\}$ by $[n]$. Finally, if $\mathbb{F}_0$ is a subfield of a field $\mathbb{F}$ and $A$ is an $\mathbb{F}$ matrix, we write row$_{\mathbb{F}_0}(A)$ for the vector space containing all linear combinations of the rows of $A$ with coefficients in $\mathbb{F}_0$. We define col$_{\mathbb{F}_0}(A)$ similarly.

The versions of connectivity we consider are all ‘vertical’; for $k \in \mathbb{Z}^+ \cup \{\infty\}$ a set $A \subseteq E(M)$ is vertically $k$-separating in $M$ if $\lambda_M(A) < k$ and $\min(r_M(A), r(M \setminus A)) \geq k$, and $M$ is vertically $k$-connected if $M$ has no vertically $k'$-separating subsets for any $k' < k$. $M$ is round if it is vertically $\infty$-connected; for example cliques, projective geometries and non-binary affine geometries are round. A matroid $M$ is vertically $k$-connected if and only if its simplification is vertically $k$-connected.
Moreover if $M$ is vertically $k$-connected then $M/e$ is vertically $(k - 1)$-connected for each $e \in E(M)$; in particular if $M$ is round then so is $M/e$. We will use the following slight strengthening of a well-known result on connectivity; see [8, Theorem 8.5.7].

**Theorem 2.1** (Tutte’s Linking Theorem). Let $M$ be a matroid and $A, B \subseteq E(M)$ be disjoint sets. There is a minor $N$ of $M$ so that $E(N) = A \cup B$, $N|A = M|A$, $N|B = M|B$ and $\lambda_N(A) = \kappa_M(A, B)$.

To avoid complications arising from inequivalent representations, we will often consider matroids defined by a representation rather than axiomatically. If $\mathbb{F}$ is a field, then an $\mathbb{F}$-represented matroid on ground set $E$ is a pair $M = (U, E)$, where $U$ is a subspace of $\mathbb{F}^E$. This represented matroid has rank function given by $r_M(X) = \dim(U[X])$ for each $X \subseteq E$, where $U[X]$ is the projection of $U$ onto $\mathbb{F}^X$. Where confusion might arise, we refer to a matroid defined in the usual way as an abstract matroid; if $M$ is an $\mathbb{F}$-represented matroid then we write $\hat{M}$ for the abstract matroid with the same rank function as $M$.

Given a matrix $A \in \mathbb{F}^{X \times E}$, we write $M(A)$ for the $\mathbb{F}$-represented matroid $(\text{row}(A), E)$ and $\hat{M}(A)$ for the associated abstract matroid; here $A$ is an $\mathbb{F}$-representation of $M(A)$. We also need to formalize deletion and contraction in this context; given an $\mathbb{F}$-representation $A$ of an $\mathbb{F}$-represented matroid $M$ and a set $X \subseteq E(M)$, we write $M \setminus X$ for the $\mathbb{F}$-represented matroid $M(A[E(M) - X])$. It is easiest to define contraction in terms of duality; if $M = (U, E)$ is an $\mathbb{F}$-represented matroid then let $M^* = (U^\perp, E)$, where $U^\perp = \{v \in \mathbb{F}^E : \langle v, u \rangle = 0 \text{ for all } u \in U\}$, and $M/X = (M^* \setminus X)^*$. Given a particular representation $A$, this is equivalent to the usual matrix interpretation of contraction where we row-reduce and take a submatrix of $A$. We extend these definitions to define a minor and restriction of an $\mathbb{F}$-represented matroid, as well as extending all other usual matroidal notions such as connectivity.

If $\mathbb{F}_0$ is a subfield of $\mathbb{F}$, then two $\mathbb{F}$-matrices $A_1, A_2$ are $\mathbb{F}_0$-row-equivalent if one can be obtained from the other by elementary row-operations only involving coefficients in $\mathbb{F}_0$. Furthermore, the matrices $A_1, A_2$ are $\mathbb{F}_0$-projectively equivalent if there is a matrix $A'_1$ that is $\mathbb{F}_0$-row-equivalent to $A_1$ that can be obtained from $A_2$ by scaling columns by nonzero elements of $\mathbb{F}_0$. We also say that the $\mathbb{F}$-represented matroids $M(A_1)$ and $M(A_2)$ are $\mathbb{F}_0$-projectively equivalent. If $\mathbb{F}_0 = \mathbb{F}$ then we just say the matrices or represented matroids are projectively equivalent, and write $A_1 \approx A_2$ and $M(A_1) \approx M(A_2)$. It is clear that if $M \approx M'$ then $\hat{M} = \hat{M}'$. For each integer $n$, let $\mathcal{PG}(n - 1, q)$ denote the set of GF($q$)-matrices $G$ with row-set $[n]$ satisfying $\hat{M}(G) \cong \mathcal{PG}(n - 1, q)$.
3. Algebra

We frequently consider an extension field $F$ of a field $F_0$; our main theorem applies just when $F_0 = GF(q)$ and $F = GF(q^2)$, but some lemmas apply for arbitrary $F_0$. When the extension has degree 2 with $F = F_0(\omega)$, we often use the fact that $F$ is a dimension-2 vector space over $F_0$ with basis $\{1, \omega\}$. We require a few lemmas relating $F_0$ and $F$ in various contexts; the first is proved in \[\square\].

**Lemma 3.1.** Let $n \geq 3$ be an integer, $q$ be a prime power, and $F$ be a field with a $GF(q)$-subfield. If $A$ is an $F$-matrix with $M(A) \cong PG(n - 1, q)$, then $A$ is projectively equivalent to a $GF(q)$-matrix.

We will apply the next lemma in the case where $j = 2$ and $h = 3$.

**Lemma 3.2.** Let $F = F_0(\omega)$ be a degree-2 extension field of a field $F_0$ and let $j, h, t \in \mathbb{Z}^+$ satisfy $2j > h$ and $j, h \leq t$. If $V$ is an $h$-dimensional subspace of $F_0^t$ and $U$ is a $j$-dimensional subspace of $F^t$ such that $U \subseteq \text{span}_F(V)$, then $U \cap V$ is nontrivial.

**Proof.** Let $\{b_1, \ldots, b_h\}$ be a basis for $V$ and let $W = \text{span}_F(V)$, noting that each $w \in W$ is expressible in the form $\sum_{i=1}^h (\lambda_i + \omega \mu_i)b_i$ for some unique $\lambda, \mu \in F_0^h$. Let $\varphi : W \to F_0^h$ be the invertible linear transformation defined by $\varphi \left( \sum_{i=1}^h (\lambda_i + \omega \mu_i)b_i \right) = (\lambda_1, \ldots, \lambda_h, \mu_1, \ldots, \mu_h)$.

Now $\varphi(U)$ and $\varphi(V)$ are subspaces of $F_0^h$ with $\text{dim}(\varphi(U)) = 2j$ and $\text{dim}(\varphi(V)) = h$, so $\text{dim}(\varphi(U) \cap \varphi(V)) = 2j + h - 2h > 0$. Therefore $U \cap V$ is nontrivial, as required. $\square$

**Lemma 3.3.** Let $F_0$ be a field and $F = F_0(\omega)$ be a degree-2 extension field of $F_0$. Let $h, d, n \in \mathbb{Z}_0^+$ satisfy $h \leq d$ and let $A, B \in F_0^{d \times n}$ be matrices such that $\text{rank}(A + \omega B) = d$. If $\text{rank}(A_B) = 2d - h$ then there is a rank-$h$ matrix $Q \in F_0^{h \times d}$ such that $Q(A + \omega B)$ is an $F_0$-matrix.

**Proof.** Let $\omega^2 = s + \omega t$ for $s, t \in F_0$. If $\text{rank}(A_B) = 2d - h$ then there are matrices $Q_1, Q_2 \in F_0^{h \times d}$ such that $(Q_1|Q_2)_B = Q_1A + Q_2B = 0$ and $\text{rank}(Q_1|Q_2) = h$. Let $Q = (\omega - t)Q_1 + Q_2$; we have $Q(A + \omega B) = (Q_2A - tQ_1A + sQ_1B) + \omega(Q_1A + Q_2B)$ which is an $F_0$-matrix.

It remains to show that $\text{rank}(Q) = h$. If not, then there are row vectors $x, y \in F_0^h$ such that $x + \omega y \neq 0$ and $(x + \omega y)Q = 0$. This gives $(xQ_2 - txQ_1 + syQ_1) + \omega(xQ_1 + yQ_2) = 0$, implying that

\begin{equation}
\begin{pmatrix}
-t & s \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
Q_1
+ \begin{pmatrix}
x \\
y
\end{pmatrix}
Q_2 = 0.
\end{equation}

Note that the matrix $J = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$ satisfies $(\begin{smallmatrix} 0 & s \\ t & 0 \end{smallmatrix})J = J(\begin{smallmatrix} 0 & -t \\ -1 & 0 \end{smallmatrix})$. Set $(u_v) = J(\begin{smallmatrix} 0 \\ v \end{smallmatrix})Q_1; we will argue that $u + \omega v \neq 0$ and $(u + \omega v)(A + \omega B) = 0$,
which contradicts rank$(A + \omega B) = d$. If $u + \omega v = 0$, then \(\begin{pmatrix} u \\ v \end{pmatrix} = 0\) and, since $J$ is nonsingular, \(\begin{pmatrix} * \\ v \end{pmatrix}Q_1 = 0\). This implies $xQ_1 = yQ_1 = 0$, which together with (1) and the fact that rank$(Q_1|Q_2) = h$ yields \(\begin{pmatrix} * \\ v \end{pmatrix} = 0\), which is not the case. Therefore \(\begin{pmatrix} u \\ v \end{pmatrix} \neq 0\). We have \((u + \omega v)(A + \omega B) = (uA + svB) + \omega(uB + vA + tvB) = \left(\begin{pmatrix} 1 \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix}B\right)\. Now

\[
\begin{pmatrix} u \\ v \end{pmatrix}A + \begin{pmatrix} 0 \\ s \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}B = J\begin{pmatrix} x \\ y \end{pmatrix}Q_1A + \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix}J\begin{pmatrix} x \\ y \end{pmatrix}Q_1B
\]

\[
= J\begin{pmatrix} x \\ y \end{pmatrix}Q_1A + \begin{pmatrix} t \\ -s \\ 0 \end{pmatrix}J\begin{pmatrix} x \\ y \end{pmatrix}Q_1B
\]

\[
= -J\begin{pmatrix} x \\ y \end{pmatrix}Q_1B + \begin{pmatrix} -t \\ s \\ 0 \end{pmatrix}J\begin{pmatrix} x \\ y \end{pmatrix}Q_1B,
\]

since $Q_1A = -Q_2B$. Now combining the above with (1) we see that

\[\begin{pmatrix} u \\ v \end{pmatrix}(A + \omega B) = 0,\]

contradicting the fact that rank$(A + \omega B) = d$ and $u + \omega v \neq 0$.

\[\square\]

The above lemma has the following as a straightforward corollary.

**Lemma 3.4.** Let $\mathbb{F} = \mathbb{F}_2(\omega)$ be a degree-2 extension field of a field $\mathbb{F}_0$. Let $h, d, m, n \in \mathbb{Z}_+^*$ satisfy $0 \leq h \leq d \leq n$ and $A, B \in \mathbb{F}_0^{d \times n}$ and $P \in \mathbb{F}_0^{m \times n}$ be such that rank$(A + \omega B) = m + d$, rank$(P) = m$ and rank$(A' + \omega B') \leq m + 2d - h$. There exist matrices $A', B' \in \mathbb{F}_0^{d \times n}$ such that $(A + \omega B)$ and $(A' + \omega B')$ are row-equivalent and $B'$ has $h$ zero rows.

4. **Examples**

We now investigate the two classes of GF$(q)$-regular matroids from our main theorem. We define them differently from in the introduction in order to prove that they are both well-defined and GF$(q)$-regular. We will use the fact that projective geometries are modular; that is, that every pair of flats $F_1, F_2$ satisfies $r(F_1 \cap F_2) = r(F_1) + r(F_2) - r(F_1 \cup F_2)$.

Let $\mathbb{F}$ be a field with a GF$(q)$-subfield, $n \geq 3$ be an integer, $A \in PG(n - 1, q)$ and $N = M(A) \cong PG(n - 1, q)$. Let $L_0$ be a line of $N$ and $v \in \text{col}_E(A[L_0])$ be not parallel to any column of $A[L_0]$. Let $f \in E(N) \setminus N$ and $L$ be the collection of lines of $\text{cl}_N(L_0 \cup \{f\})$ not containing $f$, noting that $|L| = q^2$. For each $L \in L$, let $v_L$ be a nonzero vector in the rank-1 subspace $\text{col}_E(A_L) \cap \text{col}_E(v|A[f])$. Let $X = \{x_L : L \in L\}$ be a $q^2$-element set and let $\overline{A} \in \mathbb{F}[n \times (E(N) \cup X)]$ be the matrix so that $\overline{A}[E(N)] = A$ and $\overline{A}[x_L] = v_L$ for each $L \in L$. 


Lemma 4.1. The matroid $\tilde{M}(\overline{A})$ is determined up to isomorphism by the choice of $n$ and $q$.

Proof. Let $M = \tilde{M}(\overline{A})$. We have $M \setminus X = N \cong \text{PG}(n-1, q)$. Let $\mathcal{F}_N$ be the set of cyclic flats of $N$ and $\mathcal{F}_M$ be that of $M$. Let $P = \text{cl}_N(L_0 \cup \{f\})$. Note that every pair of lines of $P$ intersect. It is easy to check the following claim:

4.1.1.

$$\mathcal{F}_M = \{F : F \in \mathcal{F}_N, |F \cap P| \leq 1\}$$
$$\cup \{F \cup X : F \in \mathcal{F}_N, F \cap P = \{f\}\}$$
$$\cup \{F \cup \{x_L\} : F \in \mathcal{F}_N, F \cap P = L \in \mathcal{L}\}$$
$$\cup \{F : F \in \mathcal{F}_N, r_M(F \cap P) = 2, F \cap P \notin \mathcal{L}\}$$
$$\cup \{F \cup X : F \in \mathcal{F}_N, P \subseteq F\}.$$  

Since a matroid is determined by its collection of cyclic flats, the matroid $\tilde{M}(\overline{A})$ is therefore determined, for a given $n$ and $q$, by the naming of elements in $X$ and the choice of $N, P$ and $f$. There is only one choice for $N$ up to isomorphism, and the lemma now follows from the fact that the Aut($\text{PG}(n-1, q)$) acts transitively on pairs $(P, f)$, where $P$ is a plane containing $f$. □

We write $\overline{\text{PG}}(n-1, q)$ for any matroid isomorphic to $M(\overline{A})$. Note that $M = \overline{\text{PG}}(n-1, q)$ arises from $N = \text{PG}(n-1, q)$ by adding $q^2$ new points on a line, spanned by a plane $P$ of $M$ and spanning a single point of $P$. The following is immediate from the definition and the previous lemma.

Lemma 4.2. The matroid $\overline{\text{PG}}(n-1, q)$ is GF($q$)-regular.

We now turn to our second class, which is simpler to analyse. Let $\mathbb{F}$ be a field with a GF($q$)-subfield and let $n \geq 2$. Let $B \in \mathcal{PG}(n, q)$ and $N = \tilde{M}(B)$. Let $L_0$ be a line of $N$ and $v \in \text{col}_F(B[L_0])$ be a nonzero vector, not parallel to any column of $B[L_0]$. Let $e \notin E(N)$ and $B^+ \in \mathbb{F}^{[n+1] \times (E(N) \cup \{e\})}$ be such that $B^+[E(N)] = B$ and $B^+[e] = v$.

By modularity of $N$, the matroid $\tilde{M}(B^+)$ is isomorphic to the principal extension of $L_0$ in $N$ by the element $e$, and is therefore determined up to isomorphism by $n$ and $q$ (due to transitivity of Aut($\text{PG}(n, q)$) on its set of lines). We write $\text{PG}(n-1, q)$ for any matroid isomorphic to the rank-$n$ matroid $\text{si}(\tilde{M}(B^+)/e)$. The following is clear by construction:

Lemma 4.3. The matroid $\text{PG}(n-1, q)$ is GF($q$)-regular.
While we have specified these matroids abstractly to emphasise their GF(q)-regularity and the fact that they are well-defined, we will only be interested in their GF(q^2)-representations. We first consider $\mathbb{P}G(n-1,q)$. The line $X$ we add is a $U_{2,q^2+1}$-restriction spanned by an element $f$ of $N$, together with an element $x_{L_0}$ that is spanned by $L_0$ but not contained in $L_0$. Since there are at most $q^2 + 1$ points on every line in $\mathbb{P}G(n-1,q^2)$, there is only one way to add the points in $X$ given a choice of $f$ and $x_{L_0}$. By choosing a basis for GF(q^2) in which $L_0$ and $f$ correspond to the first three standard basis vectors, we see that $\mathbb{P}G(n-1,q)$ has the following as a representation:

$$A(n-1,q) = \begin{pmatrix} x_{L_0} & X - \{x_{L_0}\} & E(N) \\ 1 & \alpha & \\ \omega & \omega\alpha & \\ 0 & 1 & A \\ \vdots & \vdots & \\ \end{pmatrix},$$

where $\alpha$ ranges over GF(q^2) - \{0\}, and $A \in \mathcal{P}G(n-1,q)$ is such that $A_f$ is the third standard basis vector.

Now we consider $\mathbb{P}\bar{G}(n-1,q)$. Let $B \in \mathcal{P}G(n,q)$ be a matrix containing among its columns the standard basis vectors $b_1,\ldots,b_{n+1} \in \text{GF}(q)^{n+1}$. If we choose $L_0$ to be the line spanned by $b_1$ and $b_2$ and $v$ to be the vector $b_1 - \omega b_2$, the matroid $\mathbb{P}\bar{G}(n-1,q)$, obtained by appending $v$ to $B$ and contracting the corresponding element, has the following representation:

$$\bar{A}(n-1,q) = \begin{pmatrix} (0 + 0 \omega)j & (1 + 0 \omega)j & \ldots & (s + t\omega)j & \ldots \\ 1 & A \\ \omega & \omega A \\ 0 & 1 & A \\ \vdots & \vdots & \\ \end{pmatrix},$$

where $A \in \mathcal{P}G(n-2,q)$, $j = (1,\ldots,1)$ denotes the all-ones vector with $q^n - 1 \over q-1$ entries, and $s$ and $t$ range over GF(q). Note that every vector in GF(q^2)^n with all but the first entry in GF(q) is parallel to a column of $\bar{A}(n-1,q)$.

We have defined $\mathbb{P}\bar{G}(n-1,q)$ and $\mathbb{P}G(n-1,q)$ abstractly, not as GF(q^2)-represented matroids. When we refer to the associated GF(q^2)-represented matroids we will write $M(\bar{A}(n-1,q))$ and $M(\bar{A}(n-1,q))$.

5. Non-examples

Let $F = F_0(\omega)$ be a degree-2 extension field of a field $F_0$. For a vector $w \in F^t$, we write $L(w)$ for the subspace span_{F_0}({u,v}), where $u$ and...
are the unique \( \mathbb{F}_0 \)-vectors so that \( w = u + \omega v \). Note that \( L(w) \) has dimension 2 if and only if \( w \) is not parallel to an \( \mathbb{F}_0 \)-vector.

We now define an important class of rank-3 represented matroids that will serve as obstructions to GF\((q^2)\)-regularity. Let \( \mathcal{O}(q) \) denote the set of GF\((q^2)\)-represented matroids \( M \) such that \( M \approx M(A \mid G_3) \), where the column set \( X \) of \( A \) has three elements, \( G_3 \in \mathcal{P} \mathcal{G}(2, q) \), and \( A \in \text{GF}(q^2)^{3 \times X} \) is a rank-3 matrix such that the three subspaces \( L(A_x) : x \in X \) each have dimension 2 and together have trivial intersection.

More geometrically, if \( M \in \mathcal{O}(q) \) then \( M \) is obtained by extending a projective plane \( R \) over GF\((q)\) by a three-element independent set \( X \) so that \( M \) is GF\((q^2)\)-representable and there is no point of \( R \) common to the three lines of \( R \) spanning the three points of \( X \).

**Lemma 5.1.** If \( M \in \mathcal{O}(q) \), then \( M \) is representable over a field \( \mathbb{F} \) if and only if \( \mathbb{F} \) has GF\((q^2)\) as a subfield.

**Proof.** Let \( M \in \mathcal{O}(q) \) and \( X, A, G_3 \) be defined as above. Let \( X = \{x_1, x_2, x_3\} \) and \( R = M \setminus X \), noting that \( R \cong \mathcal{P} \mathcal{G}(2, q) \). Each pair of subspaces in \( \{L(A_x) : x \in X\} \) meet in dimension 1; let \( e_i \) be the unique element of \( E(R) \) so that \( G_3[e_i] \in \cap_{j \in [3] \setminus \{i\}} (L(x_j)) \). Moreover by Lemma 3.2 each pair of columns of \( A \) spans a nonzero GF\((q)\)-vector; for each \( i \in [3] \) let \( f_i \) be the unique element of \( E(R) \) so that \( G_3[f_i] \in \text{col}(A[X - \{x_i\}]) \). Note that \( \tilde{M} \) is a simple rank-3 matroid, that \( R \cong \mathcal{P} \mathcal{G}(2, q) \), and that the subspaces \( L(A_x) : x \in X \) correspond to three lines \( L_1, L_2, L_3 \) of \( R \) so that \( x_i \in c_{L_i}(L_i) \) and \( L_1 \cap L_2 \cap L_3 = \emptyset \).

Further observe that if \( i, j \in [3] \) and \( i \neq j \), then \( f_i \notin L_j \). Since \( \tilde{M} \) is GF\((q^2)\)-representable it is also representable over all fields with a GF\((q^2)\)-subfield, so it remains to show that \( \tilde{M} \) is not representable over any other fields.

Let \( \mathbb{F} \) be a field over which \( \tilde{M} \) is representable and assume for a contradiction that \( \mathbb{F} \) does not have a GF\((q^2)\)-subfield. Since \( R \) is a minor of \( \tilde{M} \) it follows that \( \mathbb{F} \) has GF\((q)\) as a subfield. Let \( P \in \mathbb{F}^{[3] \times E(\tilde{M})} \) be a \( \mathbb{F} \)-representation of \( \tilde{M} \); by Lemma 3.1 we may assume that \( P[E(R)] \) is a GF\((q)\)-matrix and by applying further GF\((q)\)-row operations and GF\((q^2)\)-column scalings we may assume (using the fact that \( f_i \notin L_j \) for \( i \neq j \)) that \( P \) has the form

\[
P = \begin{pmatrix}
e_1 & e_2 & e_3 & x_1 & x_2 & x_3 & f_1 & f_2 & f_3 \\
1 & 0 & 0 & 0 & \alpha_2 & \alpha_3 & s_1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & s_2 & s_4 \\
0 & 0 & 1 & \alpha_1 & 1 & 0 & 1 & s_3 & s_5
\end{pmatrix},
\]
where \( \alpha_i \in \mathbb{F} - \text{GF}(q) \) for each \( i \in [3] \), \( s_1 \in \{0, 1\} \) and \( s_j \in \text{GF}(q) \) for each \( j \in [5] \). Since \( r_{\hat{M}}(x_2, x_3, f_1) = 2 \), we have \( \alpha_2 + \alpha_3 = s_1 \). The lines \( cl_{\hat{M}}(\{f_2, x_3\}) \) and \( cl_{\hat{M}}(\{x_3, f_2\}) \) both intersect \( L_2 \) at \( x_1 \), so the vectors \((0, 1 - \alpha_3s_2, -\alpha_3s_3)\) and \((0, -\alpha_2s_4, 1 - \alpha_2s_5)\) are both parallel to \((0, 1, \alpha_1)\) and thus \( \alpha_3s_3\alpha_2s_4 = (1 - \alpha_3s_2)(1 - \alpha_2s_5) \). Using \( \alpha_3 = s_1 - \alpha_2 \), we see that \( \alpha_2 \) is a zero of the function
\[
p(z) = s_3s_4z(s_1 - z) - (1 - s_2(s_1 - z))(1 - s_5z).
\]

Now \( p(z) \) is a polynomial in \( z \) with coefficients in \( \text{GF}(q) \) and degree at most 2. However, \( \alpha_2 \not\in \text{GF}(q) \) and, since \( \mathbb{F} \) has no \( \text{GF}(q^2) \)-subfield, \( \alpha_2 \) is not a zero of an irreducible quadratic over \( \text{GF}(q) \). Therefore \( p(z) \) is identically zero. We have \( 0 = p(0) = 1 - s_1s_2 \), so \( s_1s_2 = 1 \); since \( s_1 \in \{0, 1\} \) this gives \( s_1 = s_2 = 1 \). Similarly we have \( 0 = p(s_1) = s_5 - 1 \), so \( s_5 = 1 \). Therefore \( p(z) = z(1 - z)(s_3s_4 - 1) \), so \( s_3s_4 = 1 \). Let \( s_3 = t \) and \( s_4 = t^{-1} \). Since \( r_{\hat{M}}(\{x_1, x_3, f_2\}) = r_{\hat{M}}(\{x_1, x_2, f_3\}) = 2 \), we have \( \alpha_1 = t(1 + \alpha_2^{-1}) \) and \( \alpha_1 + (1 - \alpha_2)^{-1} = t \). A computation gives \( \alpha_2 = (1 + t)^{-1} \), contradicting \( \alpha_2 \not\in \text{GF}(q) \). \( \square \)

We now precisely determine the matrices \( A \) which, when appended to a matrix in \( PG(t - 1, q) \), yield a matroid with no \( \mathcal{O}(q) \)-minor; these matrices are all essentially restrictions of \( \hat{A}(t - 1, q) \) and \( \overline{A}(t - 1, q) \). We also give an alternative characterisation of these matrices in terms of the subspaces \( L(x) \) defined as above. This is equivalent to a treatment of the special case of our main theorem where \( M \) has a spanning projective geometry restriction.

**Lemma 5.2.** Let \( q \) be a prime power, \( t \geq 3 \) be an integer and \( G_i \in PG(t - 1, q) \). If \( A \in \text{GF}(q^2)^{|x| \times Y} \) and \( M = M(A \mid G_i) \) then the following are equivalent:

1. \( M \) has a minor in \( \mathcal{O}(q) \);
2. \( si(M) \) is not projectively equivalent to a restriction of either \( M(\hat{A}(t - 1, q)) \) or \( M(\overline{A}(t - 1, q)) \);
3. there exists a set \( Z \subseteq Y \), independent in \( M \), such that \(|Z| \in \{2, 3\} \) and the subspaces \( L(A_z) : z \in Z \) each have dimension 2 and have trivial intersection.

Moreover, if \( t \geq 5 \) and \( \boxed{3} \) is satisfied by a set \( Z \) of size 2, then the matroid \( M/Z \setminus (Y - Z) \) also has a minor in \( \mathcal{O}(q) \).

We call a matrix \( A \) satisfying the conditions in this lemma \( q \)-bad and if \( \boxed{3} \) holds with \(|Z| = 2 \) we call \( A \) strongly \( q \)-bad. Note that property \( \boxed{3} \), and therefore (strong) \( q \)-badness, is invariant under \( \text{GF}(q) \)-row equivalence.
Proof of Lemma 5.2: Let $b_1, \ldots, b_4$ be the standard basis vectors of $\mathbb{GF}(q)^4$. We showed in Lemmas 4.2 and 4.3 that $\overline{\mathcal{P}} \mathcal{G}(n-1, q)$ and $\overline{\mathcal{P}} \mathcal{G}(n-1, q)$ are $\mathcal{GF}(q)$-regular and in Lemma 5.1 that the matroids in $\mathcal{O}(q)$ are not, so (1) implies (3).

Suppose that (2) holds. Note that (3) and its negation are invariant under $\mathcal{GF}(q)$-row-equivalence. Let $Y' = \{y \in Y : \dim(L(A_y)) = 2\}$ and $\mathcal{L} = \{L(A_y) : y \in Y'\}$, noting that every $y \in Y - Y'$ is a loop or is parallel to some column of $G_t$, so $\text{si}(M \setminus (Y - Y')) \cong \text{si}(M)$. If there exist $z_1, z_2 \in Y'$ such that $L(A_{z_1})$ and $L(A_{z_2})$ are skew then $Z = \{z_1, z_2\}$ satisfies (3), so we may assume that $Y'$ contains no such pair.

If all subspaces in $\mathcal{L}$ have a dimension-1 subspace in common, then, by applying $\mathcal{GF}(q)$-row-operations, we may assume that this subspace is $\text{span}_{\mathcal{GF}(q)}(b_1)$. This gives a matrix representation of $\text{si}(M)$ that is, up to column scaling, a submatrix of $\tilde{A}(t-1, q)$, contradicting (2). We may therefore assume that $\cap \mathcal{L}$ is trivial.

Therefore no pair of subspaces in $\mathcal{L}$ are orthogonal but there is no dimension-1 subspace common to all subspaces in $\mathcal{L}$. It follows routinely that there is some dimension-$3$ subspace $P$ of $\mathcal{GF}(q)^4$ containing all subspaces in $\mathcal{L}$, so $r_M(Y') \leq 3$.

If $r_M(Y') \leq 2$ then there is a dimension-$2$ subspace $L_0$ of $\text{span}_{\mathcal{GF}(q^2)}(P)$ containing $A[Y']$. By Lemma 3.2, $L_0$ contains a nonzero $\mathcal{GF}(q)$-vector $v$. Let $\{v, w\}$ be a basis for $L_0$. After $\mathcal{GF}(q)$-row-operations we may assume that $\{b_1, b_2, b_3\}$ is a basis for $P$, that $v = b_4$, and that $w \in \text{cl}_{\mathcal{GF}(q^2)}(\{b_1, b_2\}) - \text{cl}_{\mathcal{GF}(q^2)}(b_2)$. Moreover, after row-scalings over $\mathcal{GF}(q^2)$ we may assume that either $w = b_1$ or $w = b_1 + \omega b_2$. Since $r_M(Y') = 2$ it follows that $\text{si}(M)$ is projectively equivalent to a restriction of $\tilde{A}(t-1, q)$ or $\overline{\tilde{A}}(t-1, q)$, contradicting (2).

If $r_M(Y') = 3$ then let $Z = \{z_1, z_2, z_3\}$ be a basis for $Y'$. Let $L_i = L(A_{z_i})$ for each $i \in \{1, 2, 3\}$. Since $r_M(Z) = 3$, the lines $L_1, L_2, L_3$ are not all equal, so we may assume that $L_1 \notin \{L_2, L_3\}$. If $L_1, L_2, L_3$ have no dimension-$1$ subspace in common then (3) holds, so we may assume that $L_1 \cap L_2 \cap L_3$ has dimension 1. Moreover we know that there is some other subspace $L_4 = L(A_{z_4}) \in \mathcal{L}$ not containing $L_1 \cap L_2 \cap L_3$, as $\cap \mathcal{L}$ is trivial. Now $L_1 \cap L_2 \cap L_4$ and $L_1 \cap L_3 \cap L_4$ are both trivial, and either $\{z_1, z_2, z_4\}$ or $\{z_1, z_3, z_4\}$ has rank 3 in $M$. Therefore (3) holds.

Finally, suppose that (3) holds. If $|Z| = 2$ then let $Z = \{z_1, z_2\}$. By applying $\mathcal{GF}(q)$-row-operations if necessary we may assume that $L(z_1) = \text{span}_{\mathcal{GF}(q)}(\{b_1, b_2\})$ and $L(z_2) = \text{span}_{\mathcal{GF}(q)}(\{b_3, b_4\})$. Let $X$ be the set of columns of $G_t$ contained in $\text{span}_{\mathcal{GF}(q)}(L(z_1) \cup L(z_2))$ and
$N = M|(X \cup \{z_1, z_2\})$. We have

$$N \approx M \begin{pmatrix} z_1 & z_2 & x \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \ldots \\ 0 & \alpha_2 \end{pmatrix},$$

for some $\alpha_1, \alpha_2 \in \text{GF}(q^2) - \text{GF}(q)$, where the matrix contains exactly one column from each parallel class in $\text{GF}(q)^4$. Therefore, $N/z_1$ is represented by a matrix having a submatrix containing as columns at least one nonzero vector from each parallel class of $\text{GF}(q)^3$, as well as columns parallel to $(0, 1, \alpha_2)^T, (-\alpha_1, 1, 0)^T$ and $(-\alpha_1, 0, 1)^T$. Restricting $N/z_1$ to this submatrix yields a matroid in $O(q)$. Moreover, if $t \geq 5$ then let $X'$ be the set of columns of $t$ contained in $\text{span}_{\text{GF}(q)}(L(z_1) \cup L(z_2) \cup \{t_5\})$ and let $N' = M|(X' \cup \{z_1, z_2\})$. It is easy to see by a similar argument to the above that $N'/\{z_1, z_2\}$, which is a restriction of $M/Z \setminus (Y - Z)$, has a spanning restriction in $O(q)$.

If (3) holds for some $Z$ of size 3 but for no 2-element subset of $Z$, then $Z$ contains three dimension-2 subspaces, all contained in a common dimension-3 subspace, with trivial intersection. This dimension-3 subspace corresponds to a plane $P$ of the spanning $\text{PG}(t - 1, q)$-restriction of $M$, and clearly $M|(P \cup Z) \in O(q)$. □

6. TANGLES

Our tool for constructing minors in $O(q)$ given a projective geometry minor (rather than a spanning restriction as in Lemma [5,2]) is the tangle. Tangles were introduced for graphs, and implicitly for matroids, by Robertson and Seymour [10] and were later extended explicitly to matroids [13]. The techniques in this section and the next follow [5].

Let $M$ be a matroid and let $\theta \in \mathbb{Z}^+$. A set $X \subseteq E(M)$ is $k$-separating in $M$ if $\lambda_M(X) < k$. A collection $\mathcal{T}$ of subsets of $E(M)$ is a tangle of order $\theta$ if

(1) Every set in $\mathcal{T}$ is $(\theta - 1)$-separating in $M$ and, for each $(\theta - 1)$-separating set $X \subseteq E(M)$, either $X \in \mathcal{T}$ or $E(M) - X \in \mathcal{T}$;

(2) if $A, B, C \in \mathcal{T}$ then $A \cup B \cup C \neq E(M)$; and

(3) $E(M) - \{e\} \notin \mathcal{T}$ for each $e \in E(M)$.

We refer to the sets in $\mathcal{T}$ as $\mathcal{T}$-small. Given a tangle of order $\theta$ on a matroid $M$ and a set $X \subseteq E(M)$, we set $\kappa_{\mathcal{T}}(X) = \theta - 1$ if $X$ is contained in no $\mathcal{T}$-small set, and $\kappa_{\mathcal{T}}(X) = \min\{\lambda_M(Z) : X \subseteq Z \in \mathcal{T}\}$ otherwise. The proof of our first lemma appears in [3]:
Lemma 6.1. If $T$ is a tangle of order $\theta$ on a matroid $M$, then $\kappa_T$ is the rank function of a rank-$(\theta - 1)$ matroid on $E(M)$.

This matroid, which we denote $M(T)$, is the tangle matroid. The next lemma is easily proved:

Lemma 6.2. If $N$ is a minor of a matroid $M$ and $T_N$ is a tangle of order $\theta$ on $N$, then $\{X \subseteq E(M) : \lambda_M(X) < \theta - 1, X \cap E(N) \in T_N\}$ is a tangle of order $\theta$ on $M$.

This tangle is the tangle on $M$ induced by $T_N$.

If $M$ is a matroid and $k$ is an integer, then we write $T_k(M)$ for the collection of $(k - 1)$-separating sets of $M$ that are neither spanning nor cospanning. For example, if $M \cong PG(n-1,q)$ and $n \geq k$, then $T_k(M)$ is simply the collection of subsets of $E(M)$ of rank at most $k - 2$. Since $3q^{n-2}-1 < q^{n-1}-1$, no three such subsets have union $E(M)$, and we easily have the following:

Lemma 6.3. If $q$ is a prime power, $n \in \mathbb{Z}^+$, and $M \cong PG(n-1,q)$, then $T_n(M)$ is a tangle of order $n$ in $M$.

If $M$ is a matroid with a $PG(n-1,q)$-minor $N$, then we write $T_n(M,N)$ for the tangle of order $n$ in $M$ induced by $T_n(N)$.

The next result is a slight variation of a lemma from [5].

Lemma 6.4. Let $k \in \mathbb{Z}^+$, let $M$ be a matroid and let $N$ be a minor of $M$ such that $T_k(N)$ is a tangle. If $X \subseteq E(M)$ is contained in a $T_k(M,N)$-small set, then there is a minor $M'$ of $M$ such that $M'|X = M|X$, $M'$ has $N$ as a minor, and $X$ is contained in a $T_k(M',N)$-small set $X'$ such that $E(M') = E(N) \cup X'$ and $\lambda_{M'}(X') = \kappa_{T_k(M',N)}(X) = \kappa_{T_k(M,N)}(X)$.

Proof. Let $b = r_{T_k(M,N)}(X)$ and let $M'$ be a minimal minor of $M$ such that $N$ is a minor of $M$, $M|X = M'|X$ and $r_{T_k(M',N)}(X) = b$. Let $T = T_k(M',N)$ and $X' = cl_{M(T)}(X)$. It remains to show that $E(M') = X' \cup E(N)$. If not, there is some $e \in E(M') - (X' \cup E(N))$. Since $cl_{M'}(X) \subseteq X'$, we know that $M|X$ is a restriction of both $M/e$ and $M \setminus e$. If $N$ is a minor of $M/e$, then by choice of $M$ we have $r_{T_k(M/e,N)}(X) \leq b - 1$. Therefore there is some set $Z \in T_k(M/e,N)$ such that $\lambda_{M'/e}(Z) \leq b - 1$ and $X \subseteq Z$. Therefore $Z \cup \{e\} \in T$ and $\lambda_{M'}(Z \cup \{e\}) \leq b$ so $r_T(X \cup \{e\}) = r_T(X)$ and $e \in cl_T(X)$, a contradiction. The case where $N$ is a minor of $M \setminus e$ is similar. \hfill $\square$

7. Using a Tangle

Our first lemma allows us to find an affine geometry restriction in a dense $GF(q)$-representable matroid $M$ after contracting a subset of
Let $x$ and for each $i$

Proof. Let $(M/C)$ such that then for each rank-$h$ $\epsilon$ GF($n \geq j < k$.

Therefore If Lemma 7.1.

the density Hales-Jewett theorem [2], but the proof of this result is much easier and we obtain a constructive bound.

**Lemma 7.1.** Let $\alpha \in \mathbb{R}^+, q$ be a prime power, and $n, h, k \in \mathbb{Z}^+$ satisfy $n \geq (2 + k)h + \log_q(2/\alpha)$ and $k \geq 2q^h(1/\alpha - 1)$. If $M$ is a rank-$r$ GF($q$)-representable matroid with $r \geq n$ and $\epsilon(M) \geq \alpha|\text{PG}(r - 1, q)|$ then for each rank-$hk$ independent set $C$ in $M$, there exists $C' \subseteq C$ such that $M/C'$ has an AG($h, q$)-restriction.

Proof. Let $(C_1, C_2, \ldots, C_k)$ be a partition of $C$ into sets of size $h$, and for each $i \in \{0, \ldots, k\}$ let $M_i = M/(C_1 \cup \ldots \cup C_i)$ and $\delta_i = \epsilon(M_i)/|\text{PG}(r(M_i) - 1, q)|$, noting that $\delta_0 \geq \alpha$ and $\delta_i \leq 1$ for each $i$.

Let $x = \frac{1}{2}q^{-h}$ and let $j$ be maximal such that $j \leq k$ and $\delta_j \geq \alpha(1 + x)j$. If $j = k$ then we have $\delta_k \geq \alpha(1 + x)^k > \alpha(1 + kx) \geq 1$, a contradiction. Therefore $j < k$, and we have $\delta_j \geq \alpha(1 + x)^j$ and $\delta_{j+1} < \alpha(1 + x)^{j+1}$.

Let $F = \text{cl}_{M_i}(C_{j+1})$ and $\mathcal{F}$ be the collection of rank-$(h + 1)$ flats of $M_j$ containing $F$; we have $\epsilon(M_{j+1}) = |\mathcal{F}|$ and $\epsilon(M_j) = \epsilon(M_j|F) + \sum_{H \in \mathcal{F}}(\epsilon(M_j|H) - \epsilon(M_j|F))$. We may assume that $M_j|H \not\cong \text{AG}(h, q)$ for each $H \in \mathcal{F}$, and therefore that $\epsilon(M_j|H) - \epsilon(M_j|F) < q^h$ for each $H \in \mathcal{F}$. Let $r = r(M_j) = n - hk$. Now

$$\alpha(1 + x)^j\frac{q^r - 1}{q - 1} \leq \epsilon(M_j)$$

$$= \epsilon(M_j|F) + \sum_{H \in \mathcal{F}}(\epsilon(M_j|H) - \epsilon(M_j|F))$$

$$\leq \frac{q^h - 1}{q - 1} + (q^h - 1)\epsilon(M_{j+1})$$

$$< \frac{q^h - 1}{q - 1} + \alpha(q^h - 1)(1 + x)^{j+1}\frac{q^{r-h} - 1}{q - 1}.$$ 

Simplifying this inequality gives

$$x(q^r - 1) + \frac{q^h - 1}{(1 + x)^{j+1}} \alpha > (1 + x)(q^h + q^{r-h} - 2),$$

and so, using $x > 0$ and $q^h \geq 2$, we have $xq^r + q^h/\alpha > q^{r-h}$. This implies that $q^r < 2q^{2h}/\alpha$, contradicting $r \geq 2h + \log_q(2/\alpha)$. \hfill \Box

We now combine the previous lemma and the machinery of tangles to show that, given a small restriction of $M$ with given ‘connectivity’ to a large projective geometry minor of $M$, we can realise the same connectivity to a projective geometry restriction in a minor of $M$. The
Lemma 7.2. Let $q$ be a prime power, let $h, a \in \mathbb{Z}^+$ satisfy $a \leq h$ and let $n = 2h(1 + q^{h+a}) + a + 2$. If $M$ is a matroid with a $\text{PG}(n-1, q)$-minor $N$ and $X \subseteq E(M)$ is a set such that $r_M(X) \leq a$ and $M \setminus X$ is $\text{GF}(q)$-representable, then there is a minor $M'$ of $M$ and a $\text{PG}(h-1, q)$-restriction $N'$ of $M'$ such that $E(M') = E(N') \cup X$, $M'|X = M|X$ and $\lambda_{M'}(X) = \kappa_{\mathcal{T}_k(M, N)}(X)$.

Proof. Let $k = 2q^{h+a}$ and $\alpha = (q^a + 1)^{-1}$, noting that $h, k, n$ and $\alpha$ satisfy the numerical conditions in Lemma 7.1. Let $b = \kappa_{\mathcal{T}_n(M, N)}(X)$. By Lemma 6.4 there is a minor $M_1$ of $M$ having $N$ as a minor and a $\mathcal{T}_n(M_1, N)$-small set $X_1$ containing $X$ such that $E(M_1) = E(N) \cup X_1$ and $\lambda_{M_1}(X_1) = \kappa_{\mathcal{T}_n(M_1, N)}(X) = b$.

Note for each independent set $C$ of $N$ that $\mathcal{T}_{n-|C|}(N/C)$ is a tangle of order $n - |C|$ on $N/C$. Let $C$ be a maximal independent set of $N \setminus (X \cap E(N))$ so that

1. $|C| \leq hk$,
2. $M_1|X = (M_1/C)|X$, and
3. $\kappa_{\mathcal{T}_{n-|C'|}(M_1/C', N/C')}(X) = b$ for all $C' \subseteq C$.

Let $M_2 = M_1/C$, $N_2 = N/C$, $\mathcal{T} = \mathcal{T}_{n-|C|}(M_2, N_2)$ and $X' = \text{cl}_M(\mathcal{T})(X)$.

7.2.1. $|C| = hk$.

Proof of claim: Suppose that $|C| \leq hk - 1$. Since $\kappa_\mathcal{T}(X') = b \leq n - hk < n - |C|$, we have $X' \in \mathcal{T}$, so $E(N_2) - X'$ is spanning in $N_2$. Further note that $r_{M_2}(X) = a < n - |C|$, let $e \in E(N_2) - X' - \text{cl}_{M_2}(X)$.

By choice of $C$ and $e$, we may assume that $X$ has rank at most $b - 1$ in $\mathcal{T}_{n-|C' \cup \{e\}|}(M_2/e, N_2/e)$ for some $C' \subseteq C$, so there is some set $Z$ such that $C' \cup \{e\} \subseteq Z$, $\lambda_{M_2/e}(Z) \leq b - 1$ and $Z \cap E(N_2/e)$ is not spanning in $N_2/e$. Therefore $(Z \cup e) \cap E(N_2)$ is not spanning in $N_2$ and $\lambda_{M_2}(Z \cup \{e\}) \leq b$. It follows that $e \in \text{cl}_{\mathcal{T}}(X) = X'$, a contradiction. \hfill \Box

Since $X_1 \cap E(N)$ is not spanning in $N$ and $N$ is round, it follows that $r_N(X_1 \cap E(N)) = \lambda_N(X_1 \cap E(N)) \leq \lambda_{M_1}(X_1) = b$. Therefore $n \leq r(M_1|E(N)) \leq n + b$. Now

\[
\epsilon(M_1 \setminus X_1) \geq \frac{q^n - 1}{q - 1} - \frac{q^b - 1}{q - 1} \\
\geq (q^b + 1)^{-1} \frac{q^{n+b} - 1}{q - 1} \\
\geq \alpha |\text{PG}(r(M_1|E(N)) - 1, q)|.
\]
The matroid $M_1|E(N)$ is a minor of $M \setminus X$ and is therefore $GF(q)$-representable. Moreover, $C$ is an $hk$-element independent subset of $E(N)$, so by Lemma 7.1, there is a set $C' \subseteq C$ such that $(M_1|E(N))/C'$ has an $AG(h,q)$-restriction $(M_1/C')|A$. Let $T' = T_{n-|C'|}(M_1/C',N/C')$. Now $N/C'$ is $GF(q)$-representable and $\epsilon((N/C')|A) = q^h$, so $r_{(N/C')|A} \geq h + 1 > b$. Therefore $\kappa_{T'}(A) \geq \kappa_{T_{n-|C'|}(N/C')} (A) \geq b$. It follows that $\kappa_{M_1/C'}(X,A) = b$, as otherwise $M_1/C'$ has a $b$-separation for which neither side is $T'$-small.

By Theorem 2.1, there is a minor $M'_1$ of $M_1/C'$ with $E(M'_1) = X \cup A$, $M'_1|X = (M_1/C')|X = M|X$, $M'_1|A = (M_1/C')|A \cong AG(h,q)$ and $\lambda_{M'_1}(X) = b$. Since $r(M'_1|A) = h + 1 > b$, there is some $e \in A - cl_{M'_1}(X)$. Contracting $e$ and simplifying yields the required minor $M'$.

Note in the above lemma that, in the special case where $M$ is round we have $\kappa_{T_{h}(M,N)}(X) = r_M(X)$; it follows that $N'$ is spanning in $M'$.

8. Augmenting Structure

We now consider a matroid $M$ and an element $e \in E(M)$ such that $si(M/e)$ is a restriction of $\widetilde{PG}(r(M) - 2, q)$ or $\overline{PG}(r(M) - 2, q)$; we essentially argue that $M$ itself either has one of these two structures, or satisfies some constructive condition certifying otherwise. Unfortunately these hypotheses and outcomes are somewhat opaque in the two lemmas that follow; Theorem 9.1 will unify them.

We consider a slight variation of contraction in this section for ease of notation. If $e$ is a nonloop of a represented matroid $M$, then we let $M/e$ denote the represented matroid $M'/e'$, where $M'$ is obtained from $M$ by extending $e$ in parallel by an element $e'$. Thus, $e$ is a loop of $M/e$, and we have $M/e = (M/e) \setminus e$ and $E(M/e) = E(M)$. Note that if $M/e \approx M(A)$ for some $\mathbb{F}$-matrix $A$, then $M \approx M(A')$ for some matrix $A'$ obtained by appending a single row to $A$.

**Lemma 8.1.** Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field $\mathbb{F}_0$. Let $M$ be a vertically $5$-connected $\mathbb{F}$-represented rank-$r$ matroid and $e$ be a nonloop of $M$ such that $M/e \approx M\left(\begin{array}{c}u_0 + \omega v_0 \\ R\end{array}\right)$ for some $u_0, v_0 \in \mathbb{F}_0^{E(M)}$ and $R \in \mathbb{F}_0^{(r-2) \times E(M)}$. Then there are matrices $P, Q \in \mathbb{F}_0^{[2] \times E(M)}$ such that $M \approx M\left(\begin{array}{c}P + \omega Q \\ R\end{array}\right)$ and either

1. there is a partition $(I, J)$ of $E(M)$ such that
   \[\text{rank}(R[I]) + \text{rank}(Q[J]) \leq 1,\]
2. or the matrix
Proof. Since $M/\parallel e \approx M^{(\omega_0+\omega_0)}$, we have $M \approx M^{(P_1+\omega Q_1)}$ for some $P_1, Q_1 \in \mathbb{F}_0^{[2] \times E(M)}$. Let $W^+$ be the matrix in (2) with $P, Q = P_1, Q_1$ and let $M^+ = M(W^+)$. Note that $M \approx M^+/X \setminus S$ and $r(M^+) = r + 2$. If (2) does not hold for $P_1, Q_1$, then there are sets $Z, K \subseteq E(M^+)$ such that $r_M(K) \geq 4$, with $S \cup X \subseteq Z \subseteq E(M^+) - K$ and $\lambda_{M^+}(Z) \leq 3$. Let $(I, J) = (E(M) \cap Z, E(M) - Z)$.

Note that $r_{M^+}(Z) \geq r_{M^+}(S) = 4$. We have $\lambda_M(I) \leq \lambda_{M^+}(Z) \leq 3$, so vertical 5-connectivity of $M$ gives $\min(r_M(I), r_M(J)) \leq 3$. But $r_M(J) \geq r_M(K) \geq 4$, so $r_M(J) \geq 3$. This gives $r_{M^+}(Z) \leq 5$ and, by vertical 5-connectivity of $M$, $r_M(J) = r$.

Note that $0 \leq r_{M^+}(J) - r_M(J) \leq r(M^+) - r(M) = 2$. We have $r = r_M(J) = \text{rank}((P_1+\omega Q_1)_R[J])$ and $r_{M^+}(J) = \text{rank}(W^+[J])$. By Lemma 3.4, $(P_1+\omega Q_1)_R[J]$ is row-equivalent to a matrix $(P_1+\omega Q_1)_R[J]$, where

$$\text{rank}(Q') = \text{rank}(W^+[J]) - \text{rank}((P_1+\omega Q_1)_R[J]) = r_{M^+}(J) - r.$$ 

Therefore $(P_1+\omega Q_1)_R$ is row-equivalent to a matrix $(P_1+\omega Q_1)_R$ where $Q[J] = Q'$. Now $M = M^{(P_1+\omega Q_1)_R}$ and

$$3 \geq \lambda_{M^+}(Z) = r_{M^+}(Z) + r_{M^+}(J) - r(M^+) = (4 + \text{rank}(R[I])) + (r + \text{rank}(Q')) - (r + 2) = 2 + \text{rank}(R[I]) + \text{rank}(Q[J])$$ 

so $\text{rank}(R[I]) + \text{rank}(Q[J]) \leq 1$. Therefore (1) holds.

Lemma 8.2. Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field $\mathbb{F}_0$. Let $M$ be a rank-$r$, vertically 9-connected $\mathbb{F}$-represented matroid and $e$ be a nonloop of $M$. If there are matrices $P_0, Q_0 \in \mathbb{F}_0^{[2] \times E(M)}$ and $R \in \mathbb{F}_0^{[3] \times E(M)}$ and a partition $(I_0, J_0)$ of $E(M)$ such that $M/\parallel e \approx M^{(P_1+\omega Q_0)_R}$, $r_{M/\parallel e}(I_0) \leq 2$, $\text{rank}(R[I_0]) \leq 1$ and $Q_0[J_0] = 0$, then there are matrices $P, Q \in \mathbb{F}_0^{[3] \times E(M)}$ such that $M \approx M^{(P_1+\omega Q_1)_R}$ and either

1. $M$ and $e$ satisfy the hypotheses of Lemma 8.1,
2. there is a partition $(I, J)$ of $E(M)$ such that $Q[J] = 0$ and $r_M(I) \leq 4$, or
(3) the matrix

\[
W^+ = \begin{pmatrix}
I_3 & 0 & -\omega I_3 & P \\
0 & I_3 & I_3 & Q \\
0 & 0 & 0 & R
\end{pmatrix}
\]

satisfies \( \kappa_{M(W^+)}(S \cup X, K) \geq 5 \) for each set \( K \subseteq E(M) \) such that \( r_M(K) \geq 5 \). (Here \( |S| = 6 \) and \( |X| = 3 \).)

**Proof.** By hypothesis, there are matrices \( P_1, Q_1 \in \mathbb{F}_0^{[3] \times E(M)} \) such that \( M \approx M(P_1 + \omega Q_1) \), where \( P_1 = \left(\begin{smallmatrix} u \\ \omega P_0 \end{smallmatrix}\right) \) and \( Q_1 = \left(\begin{smallmatrix} v \\ \omega Q_0 \end{smallmatrix}\right) \) for some vectors \( u, v \in \mathbb{F}_0^{E(M)} \). Let \( W^+ \) be the matrix in (3) with \( P, Q = P_1, Q_1 \) and let \( M^+ = M(W^+) \). As before, we have \( M \approx M^+/X \setminus S \), \( r(M^+) = r + 3 \) and we may assume that there are sets \( Z, K \subseteq E(M^+) \) with \( r_M(K) \geq 5 \) such that \( S \cup X \subseteq Z \subseteq E(M) - K \) and \( \lambda_M(Z) \leq 4 \).

Now \( \lambda_M(E(M) \cap Z) \leq \lambda_M(Z) \leq 4 \), so vertical 6-connectivity of \( M \) gives \( \min(r_M(E(M) \cap Z), r(M \setminus Z)) \leq 4 \), but \( r(M \setminus Z) \geq r_M(K) \geq 5 \), so \( r_M(E(M) \cap Z) \leq 4 \) and thus \( r_M(Z) \leq 7 \) and \( r_M(Z) \in \{6, 7\} \). Let \( F = \text{cl}_{M^+}(Z) \), let \( (I_1, J_1) = (E(M) \cap F, E(M) - F) \) and let \( (I, J) = (I_0 \cup I_1, J_0 \cup J_1) \).

We have \( r_M(I) \leq (r_{M/\{e\}}(I_0) + 1) + r_M(I_1) \leq 3 + 4 = 7 \), so by vertical 9-connectivity of \( M \) we get \( r_M(J) = r \). Therefore \( r_M(J) \geq r \). Moreover \( r_M(J_1) = r + \lambda_M(J_1) - r_M(F) \leq (r + 3) + 4 - r_M(F) = r + 7 - r_M(Z) \), so \( r_M(J_1) \in \{r, r + 1\} \). We consider the two cases separately.

If \( r_M(J_1) = r \) then \( r_M(J) = r \) and \( W^+[J] \) is a rank-\( r \) matrix with \( (r + 3) \) rows, so by Lemma 3.4 \((P_{R'}^{r + \omega Q_1})[J] \) is row-equivalent to a matrix \((P_{R'[J]}^r) \) where \( P_{R'} \in \mathbb{F}_0^{[3] \times J} \). Therefore \((P_{R}^{r + \omega Q_1}) \) is row-equivalent to a matrix \((P_{R'}^{r + \omega Q_0}) \) where \( Q[J] = 0 \). Now \( M \approx M(P_{R'}^{r + \omega Q_0}) \) and \( r_M(I) \leq r_M(Z) - 3 \leq 4 \), so (2) holds.

If \( r_M(J_1) = r + 1 \) then \( r_M(F) = 6 = r_M(S) \) so \( F = \text{cl}_{M^+}(S) \). It follows that \( R[I_1] = 0 \). Also, \( W^+[J] \) is a rank-\( (r + 1) \) matrix with \( r + 3 \) rows, so by Lemma 3.4 the matrix \((P_{R'[J_1]}^{r + \omega Q_1})[J] \) is row-equivalent to a matrix \((P_{R'[J]}^{r + \omega Q'}) \) where \( P', Q' \in \mathbb{F}_0^{[3] \times J_1} \) and \( Q'[J] \) has two zero rows. Therefore \((P_{R'}^{r + \omega Q_1}) \) is row-equivalent to a matrix \((P_{R'}^{r + \omega Q_0}) \) where \( P, Q \in \mathbb{F}_0^{[3] \times E(M)} \) and \( Q[J_1] = Q' \). Since \( R[e] = 0 \), it follows that \( M/e \approx M(P_{R'}^{r + \omega Q_0}) \) for some matrices \( P', Q' \in \mathbb{F}_0^{[2] \times E(M)} \) with \( \text{rank}(Q'[J_1]) \leq \text{rank}(Q') \leq 1 \). We may assume (by applying \( \mathbb{F}_0 \)-row operations to \( P_0' + \omega Q_0' \) if necessary) that the second row of \( Q_0'[J_1] \) is zero. Now \( R[I_1] = 0 \), so we can scale each column of \((P_{R'}^{r + \omega Q_0})[J_1] \) to have its
second entry in $\mathbb{F}_0$. This yields an matrix $(u_0 + \omega v_0) R'$ where $u_0, v_0$ are $\mathbb{F}_0$-vectors, $R'$ is an $\mathbb{F}_0$-matrix, and $M/e \approx M(u_0 + \omega v_0)$, so (1) holds.

9. The Main Theorem

By Lemma 5.1, the abstract matroids corresponding to the represented matroids in $\mathcal{O}(q)$ are not GF($q$)-regular. By Lemmas 4.2 and 4.3, restrictions of $\mathbb{P}G(r-1, q)$ and $\mathbb{P}\mathcal{G}(r-1, q)$ are GF($q$)-regular. The following result, which applies to arbitrary GF($q^2$)-represented matroids, thus has Theorem 1.1 as a corollary.

Theorem 9.1. Let $q$ be a prime power. If $M$ is a round rank-$r$ GF($q^2$)-represented matroid with a $\mathbb{P}G(12q^{12} + 19, q)$-minor and no minor in $\mathcal{O}(q)$, then si($M$) is projectively equivalent to a restriction of either $M(\hat{A}(r-1, q))$ or $M(\overline{A}(r-1, q))$.

Proof. Let $n = 12q^{12} + 20$ and $N$ be a $\mathbb{P}G(n-1, q)$-minor of $M$. Let $\mathcal{T} = \mathcal{T}_n(M, N)$.

If $N$ is spanning in $M$ then, by Lemma 3.1, we have $M \approx M(A \mid G_r)$ for some matrices $G_r \in \mathbb{P}G(r-1, q)$ and $A$, and the result follows from Lemma 5.2. We may thus assume inductively that there exists $e \in E(M)$ so that $N$ is a minor of $M/e$ and si($M/e$) is a restriction of either $\hat{\mathbb{P}}G(r-2, q)$ or $\overline{\mathbb{P}}\mathcal{G}(r-2, q)$. We consider these cases in two mutually exclusive claims.

9.1.1. If the matroid si($M/e$) is projectively equivalent to a restriction of $M(\hat{A}(r-2, q))$ then the theorem holds.

Proof of claim: The matroid $M$ is round (so is vertically 5-connected) and has a GF($q^2$)-representation projectively equivalent to a submatrix of $\hat{A}(r-2, q)$; it follows that $M$ and $e$ satisfy the hypotheses of Lemma 8.1. Define matrices $P, Q, R$ as in the conclusion of the lemma, so $M \approx M(W)$ where $W = (P + \omega Q) R$.

If outcome (1) of Lemma 8.1 holds then there is a partition $(I, J)$ of $E(M)$ so that rank($R[I]$) + rank($Q[J]$) $\leq 1$, so one of these matrices is zero and the other has rank at most 1. If $R[I] = 0$ and rank($Q[J]$) $\leq 1$ then we may perform GF($q$)-row-operations in the first two rows so that only the first row of $Q[J]$ is nonzero and then scale each column in $I$ so that the second entry is in $\{0, 1\}$; since $R[I] = 0$ it follows that si($M$) is projectively equivalent to a restriction of $M(\hat{A}(r-1, q))$, as required.

If $Q[J] = 0$ and rank($R[I]$) $\leq 1$, then let $A = W[I]$. Note that $r_M(I) \leq 3$. Since $Q[J] = 0$, if the matroid si($M(A \mid G_r)$) is projectively
equivalent to a restriction of $M(\hat{A}(r - 1, q))$ or $M(\hat{A}(r - 1, q))$ then so is $\text{si}(M)$. Otherwise, $A$ is $q$-bad (recall Section 5 for a definition). By roundness of $M$ and Lemma 7.2 applied with $a = h = 3$, there is a rank-3 minor $M'$ of $M$ with a $\text{PG}(2, q)$-restriction $N'$ so that $E(M') = E(N') \cup I$ and $M'|I = M|I$. However $M'$ is obtained from $M$ by contracting and deleting only columns in $W[J]$, so if $G_3 \in \mathcal{PG}(2, q)$ then $M' \approx M(A' \mid G_3)$ for some matrix $A'$ that is $\text{GF}(q)$-row-equivalent to $A$; the matrix $A'$ is also $q$-bad, so by Lemma 5.2 the matroid $M'$ has a minor in $\mathcal{O}(q)$.

If outcome (2) of the lemma holds then let $W^+$ be the given matrix and $M^+ = M(W^+)$, noting that $M \approx M^+/X \setminus S$ and that $W^+[S \cup X]$ is strongly $q$-bad (with $Z = X$). Let $T^+ = T_n(M^+, N)$. Since $\kappa_{M^+}(S \cup X, K) \geq 4$ for each basis or cobasis $K$ of $N$, it follows that $\kappa_{T^+}(S \cup X) = 4$ and so, by Lemma 7.2 applied with $a = 4$ and $h = 5$, $M^+$ has a minor $M'$ with a $\text{PG}(4, q)$-restriction $N'$ so that $E(M') = E(N') \cup (S \cup X)$ and $M'|I(S \cup X) = M|(S \cup X)$. Similarly to the previous case, we have $M' \approx M(B \mid G_5)$ for some $G_5 \in \mathcal{PG}(4, q)$ and some matrix $B$ that is $\text{GF}(q)$-row-equivalent to $W^+[S \cup X]$ and hence strongly $q$-bad. By Lemma 5.2, the matroid $M'/X \setminus S$, which is a minor of $M$, has a minor in $\mathcal{O}(q)$, again a contradiction. \[\square\]

9.1.2. If the matroid $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\hat{A}(r - 2, q))$ but not to a restriction of $M(\hat{A}(r - 2, q))$ then the theorem holds.

\begin{proof}
Since $M$ it is vertically 9-connected. Since $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\hat{A}(r - 2, q))$, it is easy to see that $M$ and $e$ satisfy the hypotheses of Lemma 8.2. (The required partition $(I_0, J_0)$ is induced by the line $L_0$ and its complement in the column set of $\hat{A}(r - 2, q)$.) If outcome (1) of the lemma holds then $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\hat{A}(r - 2, q))$, a contradiction. Therefore (2) or (3) holds. Let $M \approx M(W)$ where $W = \left( ^{P+\omega Q}_{R} \right)$ as in the lemma.

Suppose that (2) holds, and let $(I, J)$ be the associated partition of $E(M)$. If $\text{si}(M(W[I] \mid G_r))$ is projectively equivalent to a restriction of $M(\hat{A}(r - 1, q))$ or $M(\hat{A}(r - 1, q))$ then, as $W[J]$ is a $\text{GF}(q)$-matrix, so is $\text{si}(M)$. Therefore we may assume that this is not the case, so $W[I]$ is $q$-bad. By roundness of $M$ we have $\kappa_T(I) = r_M(I) \leq 4$, so Lemma 7.2 with $a = h = 4$ gives a rank-4 minor $M'$ of $M$ with a $\text{PG}(3, q)$-restriction $N'$ satisfying $E(M') = E(N') \cup I$ and $M'|I = M|I$. Now $E(M) - E(M') \subseteq J$ and so $M' \approx M(B \mid G_4)$ for some $G_4 \in \mathcal{PG}(3, q)$ and some matrix $B$ that is $\text{GF}(q)$-row-equivalent to
\[ W[I] \] and hence \( q \)-bad. Lemma 5.2 implies that \( M' \) has a minor in \( \mathcal{O}(q) \), a contradiction.

Finally, suppose that (3) holds. Let \( W^+ \) be the matrix given and let \( M = M(W^+) \), noting that \( M = M'/X \setminus S \). Let \( T^+ = T_n(M^+, N) \).

Since \( \kappa_{M^+}(S \cup X, K) \geq 5 \) for each basis or cobasis \( K \) of \( N \), we have \( \kappa_{T^+}(S \cup X) \geq 5 \). By Lemma 7.2 with \( a = h = 6 \) there is a minor \( M' \) of \( M^+ \) and a \( \text{PG}(5, q) \)-restriction \( N' \) of \( M' \) so that \( E(M') = E(N' \cup X \cup S) \), \( M'|(X \cup S) = M|(X \cup S) \) and \( \lambda_{M'}(X \cup S) \geq 5 \), from which it follows that \( 6 \leq r(M') \leq 7 \).

Since \( W^+[E(M)] \) is a \( \text{GF}(q) \)-matrix, we have \( M' \approx M(B \mid G) \), where \( B \) is obtained by appending a row of zeroes above \( W^+[S \cup X] \) and \( G \) is a \( \text{GF}(q) \)-representation of \( N' \approx \text{PG}(5, q) \) with 7 rows. (If \( r(M') = 6 \) then the first row of \( G \) is also zero). Let \( v_0, \ldots, v_6 \) denote the row vectors of \( G \), so \( M'/X \setminus S \approx M(W') \), where

\[
W' = \begin{pmatrix}
v_0 \\
v_1 + \omega v_4 \\
v_2 + \omega v_5 \\
v_3 + \omega v_6
\end{pmatrix}.
\]

For each \( i \in \{0, \ldots, 6\} \) let \( G^i \) be the matrix obtained by removing the \( i \)th row of \( G \). Since \( \tilde{M}(G) \cong \text{PG}(5, q) \), there is some \( i \in \{0, \ldots, 6\} \) so that \( \tilde{M}(G^i) \cong \text{PG}(5, q) \). Furthermore, unless \( v_0 = 0 \) we may choose \( i \) to be nonzero. If \( v_0 = 0 \) then, since \( \tilde{M}(G^0) \cong \text{PG}(5, q) \), every vector in \( \text{GF}(q^2)^4 \) with first component zero is a \( \text{GF}(q) \)-multiple of some column of \( W' \), so \( \text{si}(M(W')) \cong \text{PG}(2, q^2) \) and \( M'/X \setminus S \) clearly has a restriction in \( \mathcal{O}(q) \), a contradiction.

Otherwise, we can choose \( i \) nonzero such that \( \tilde{M}(G^i) \cong \text{PG}(5, q) \). We will suppose that \( i = 6 \); the other cases are similar. Since \( G^6 \) contains a column from every parallel class in \( \text{GF}(q)^5 \), there is some \( f \in E(N') \) so that \( G^6[f] \) has all entries zero except its \( v_3 \)-entry which is nonzero. Therefore \( W'[f] \) has all entries zero except its last entry which is nonzero. Now consider a representation \( W'' \) of \( M(W')/f \) given by removing the \( f \)-column and last row from \( W' \). Since the matrix with rows \( v_0, v_1, v_2, v_4, v_5 \) has a column in every parallel class in \( \text{GF}(q)^5 \), it follows that \( W'' \) contains a column from every parallel class in \( \text{GF}(q^2)^3 \), and so \( \text{si}(M(W'')) \cong \text{PG}(2, q^2) \) and \( M(W'') \) has a restriction in \( \mathcal{O}(q) \), a contradiction.

The result now follows from the two claims. \( \square \)
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