MATROIDS DENSER THAN A PROJECTIVE GEOMETRY

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Abstract. The *growth-rate function* for a minor-closed class $\mathcal{M}$ of matroids is the function h where, for each non-negative integer r, $h(r)$ is the maximum number of elements of a simple matroid in $\mathcal{M}$ with rank at most r. The Growth-Rate Theorem of Geelen, Kabell, Kung, and Whittle shows, essentially, that the growth-rate function is always either linear, quadratic, exponential with some prime power q as the base, or infinite. Morover, if the growth-rate function is exponential with base q, then the class contains all $\mathrm{GF}(q)$-representable matroids, and so $h(r) \ge \frac{q^r-1}{q-1}$ for each r. We characterise the classes that satisfy $h(r) = \frac{q^r-1}{q-1}$ for all sufficiently large r. As a consequence, we determine the eventual value of the growth rate function for most classes defined by excluding lines, free spikes and/or free swirls.

1. Introduction

The *principal extension* of a flat F in a matroid M by an element $e \notin E(M)$ is the matroid M' such that $M = M' \backslash e$ and F is the unique minimal flat of M for which $e \in \mathrm{cl}_{M'}(F)$. We write $\widehat{\mathrm{PG}}(n-1, q; k)$ for the principal extension of $\mathrm{PG}(n-1, q)$ by a rank-k flat. We prove the following:

Theorem 1.1. *Let q be a prime power and let $\ell \ge 2$ and $n \ge 2$ be integers. If M is a simple matroid with $|M| > |\mathrm{PG}(r(M)-1, q)|$ and $r(M)$ is sufficiently large, then M has a minor isomorphic to $U_{2,\ell+2}$, $\widehat{\mathrm{PG}}(n-1, q; 2), \widehat{\mathrm{PG}}(n-1, q; n)$, or $\mathrm{PG}(n-1, q')$ for some $q' > q$.*

This result first appeared in [6] and essentially follows from material in [3], but our proof is much shorter due to the use of the matroidal density Hales-Jewett theorem [4].

Date: September 2, 2014.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. matroids, growth rates.

This research was partially supported by a grant from the Office of Naval Research [N00014-12-1-0031].

Theorem 1.1 has several corollaries related to the growth rate functions of minor-closed classes. For a nonempty minor-closed class of matroids $\mathcal{M}$, the *growth rate function* $h_{\mathcal{M}}(n) : \mathbb{Z}_0^+ \to \mathbb{Z} \cup \{\infty\}$ is the function whose value at each integer n is the maximum number of elements in a simple matroid $M \in \mathcal{M}$ with $r(M) \le n$. Clearly $h_{\mathcal{M}}(n) = \infty$ for all $n \ge 2$ if $\mathcal{M}$ contains all simple rank-2 matroids; in all other cases, growth rate functions are quite tightly controlled by a theorem of Geelen, Kabell, Kung and Whittle:

Theorem 1.2 (Growth rate theorem)**.** *Let $\mathcal{M}$ be a nonempty minor-closed class of matroids not containing all simple rank-2 matroids. There exists $c \in \mathbb{R}$ such that either:*

(1) $h_{\mathcal{M}}(n) \le cn$ for all n,
(2) $\binom{n+1}{2} \le h_{\mathcal{M}}(n) \le cn^2$ for all n, and $\mathcal{M}$ contains all graphic matroids, or
(3) there is a prime power q so that $\frac{q^n-1}{q-1} \le h_{\mathcal{M}}(n) \le cq^n$ for all n, and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids.

Our main result thus applies to the densest matroids in all class of type (3) for which the lower bound $h_{\mathcal{M}}(n) \ge \frac{q^n-1}{q-1}$ does not eventually hold with equality.

Minor-closed classes. We now give a version of our main theorem in terms of minor-closed classes, and state several corollaries. For each prime power q, let $\mathcal{L}(q)$ denote the class of $\mathrm{GF}(q)$-representable matroids. Let $\mathcal{L}^\circ(q)$ denote the closure under minors and isomorphism of $\{\widehat{\mathrm{PG}}(n-1, q; n) : n \ge 2\}$. Let $\mathcal{L}^\lambda(q)$ denote the closure under minors and isomorphism of $\{\widehat{\mathrm{PG}}(n-1, q; 2) : n \ge 2\}$. Our main theorem can be restated as follows:

Theorem 1.3. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed-class of matroids such that $\frac{q^n-1}{q-1} < h_{\mathcal{M}}(n) < \infty$ for infinitely many n, then $\mathcal{M}$ contains $\mathcal{L}^\circ(q)$, $\mathcal{L}^\lambda(q)$ or $\mathcal{L}(q')$ for some $q' > q$.*

One can easily determine the growth rate functions of $\mathcal{L}^\circ(q)$ and $\mathcal{L}^\lambda(q)$; we have $h_{\mathcal{L}^\circ(q)}(n) = \frac{q^{n+1}-1}{q-1}$ and $h_{\mathcal{L}^\lambda(q)}(n) = \frac{q^{n+1}-1}{q-1} - q$ for all $n \ge 2$. For any $q' > q$, the growth rate function of $\mathcal{L}(q')$ dominates both these functions for large n, so the following is immediate:

Theorem 1.4. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids so that $h_{\mathcal{M}}(n) > \frac{q^n-1}{q-1}$ for infinitely many n, then $h_{\mathcal{M}}(n) \ge \frac{q^{n+1}-1}{q-1} - q$ for all sufficiently large n.*

Let $\ell \geq 2$ be an integer and $\mathcal{U}(\ell)$ denote the class of matroids with no $U_{2,\ell+2}$-minor. Our next corollary is the main theorem of [3].

Theorem 1.5. *$h_{\mathcal{U}(\ell)}(n) = \frac{q^n-1}{q-1}$ for all sufficiently large n, where q is the largest prime power not exceeding ℓ.*

Let Λ_k denote the rank-k free spike (see [2] for a definition); the next corollary determines the eventual growth rate function for any class defined by excluding a free spike and a line:

Theorem 1.6. *Let $\ell \geq 2$ and $k \geq 3$ be integers. If $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$- or Λ_k-minor, then $h_{\mathcal{M}}(n) = \frac{p^n-1}{p-1}$ for all sufficiently large n, where p is the largest prime satisfying $p \leq \min(\ell, k+1)$.*

Let Δ_k denote the rank-k free swirl (again, see [2]). We do not obtain a complete version of Theorem 1.6 for swirls, but still obtain a result in a large range of cases. A *Mersenne prime* is a prime number of the form $2^p - 1$ where p is also prime.

Theorem 1.7. *Let 2^p-1 and $2^{p'}-1$ be consecutive Mersenne primes, and let k and ℓ be integers for which $2^p \leq \ell < \min(2^{2p}+2^p, 2^{p'})$ and $k \geq \max(3, 2^p-2)$. If $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$- or Δ_k-minor, then $h_{\mathcal{M}}(n) = \frac{2^{pn}-1}{2^p-1}$ for all sufficiently large n.*

If $p' > 2p$, there is a range of values of ℓ to which the above theorem does not apply. This does occur (for example, when $(p, p') = (127, 521)$) and in fact, the growth rate function for $\mathcal{M}$ can take a different eventual form for such an ℓ; we discuss this in Section 3.

For excluding both a free spike and a free swirl, we get a nice result:

Theorem 1.8. *Let $\ell \geq 3$ and $k \geq 3$ be integers. If $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$-, Λ_k- or Δ_k-minor, then $h_{\mathcal{M}}(n) = \frac{1}{2}(3^n-1)$ for all sufficiently large n.*

2. The Main Result

In this section we prove Theorem 1.1. We use the notation of Oxley [7], writing $\varepsilon(M)$ for the number of points in a matroid M. The following theorem from [4] is our main technical tool:

Theorem 2.1 (Matroidal density Hales-Jewett theorem)**.** *There is a function $f : \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z}$ so that, for every positive real number α, every prime power q and for all integers $\ell \geq 2$ and $t \geq 2$, if $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) \geq \alpha q^{r(M)}$ and $r(M) \geq f(\ell, n, q, \alpha)$, then M has an $\mathrm{AG}(n-1, q)$-restriction or a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$.*

For an integer $q \geq 2$, a matroid M is *q-dense* if $\varepsilon(M) > \frac{q^{r(M)}-1}{q-1}$. We prove an easy lemma showing when q-density is lost by contraction:

Lemma 2.2. *Let $q \geq 2$ be an integer. If M is a q-dense matroid and $e \in E(M)$, then either M/e is q-dense, or M has a $U_{2,q+2}$-restriction containing e.*

Proof. We may assume that M is simple. If $|L| \geq q+2$ for some line L through e, then M has a $U_{2,q+2}$-restriction containing e. Otherwise, no line through e contains $q+2$ points, so each point of M/e contains at most q elements of M. Therefore $\varepsilon(M/e) \geq q^{-1}\varepsilon(M \backslash e) > \frac{q^{r(M)-1}-1}{q-1}$ and $r(M/e) = r(M) - 1$, so M/e is q-dense. □

A simple induction now gives a corollary originally due to Kung [5]:

Corollary 2.3. *If $\ell \geq 2$ and $M \in \mathcal{U}(\ell)$ then $\varepsilon(M) \leq \frac{\ell^{r(M)}-1}{\ell-1}$.*

We now reduce Theorem 1.1 to a case where all cocircuits are large:

Lemma 2.4. *Let $t, \ell \geq 2$ be integers and let q be a prime power. If $M \in \mathcal{U}(\ell)$ is a q-dense matroid so that $(\sqrt{5}-1)^{r(M)-1} \geq \ell^{t-1}$, then M has a q-dense restriction M_0 such that $r(M_0) \geq t$ and every cocircuit of M_0 has rank at least $r(M_0) - 1$.*

Proof. Let $\varphi = \frac{1}{2}(1+\sqrt{5})$. Let $r = r(M)$, and let M_0 be a minimal restriction of M so that $\varepsilon(M_0) > \varphi^{r(M_0)-r}\frac{q^r-1}{q-1}$. Let $r_0 = r(M_0)$. Since $(\varphi/q)^{r_0-r} \geq 1$ and $\varphi^{r_0-r} \leq 1$, we have

$$\varepsilon(M_0) > \tfrac{1}{q-1}\left(\varphi^{r_0-r}q^r - \varphi^{r_0-r}\right) \geq \tfrac{q^{r_0}-1}{q-1},$$

so M_0 is q-dense. Moreover,

$$\varepsilon(M_0) > \varphi^{r_0-r}q^{r-1} \geq \varphi^{1-r}2^{r-1} = (\sqrt{5}-1)^{r-1} \geq \ell^{t-1} > \tfrac{\ell^{t-1}-1}{\ell-1},$$

so $r(M_0) \geq t$ by Corollary 2.3. Finally, if M_0 had a cocircuit C of rank at most $r(M_0) - 2$, minimality would give

$$\varepsilon(M_0) = \varepsilon(M_0|C) + \varepsilon(M_0 \backslash C) \leq (\varphi^{-2} + \varphi^{-1})\varphi^{r_0-r}\tfrac{q^r-1}{q-1};$$

since $\varphi^{-2} + \varphi^{-1} = 1$, this contradicts $\varepsilon(M_0) > \varphi^{r_0-r}\frac{q^r-1}{q-1}$. □

We now prove the lemma that gives one of two unavoidable minors in an arbitrary non-representable extension of a large projective geometry:

Lemma 2.5. *Let q be a prime power and $m \geq 2$ be an integer. If M is a non-$\mathrm{GF}(q)$-representable extension of $\mathrm{PG}(2m-1,q)$, then M has a minor isomorphic to $\widehat{\mathrm{PG}}(m-1,q;2)$ or $\widehat{\mathrm{PG}}(m-1,q;m)$.*

Proof. Since every flat in a projective geometry is modular, we know that M is a principal extension of some flat F of $\mathrm{PG}(2m-1,q)$. Let B be a basis for $\mathrm{PG}(n-1)$ containing a basis B_F for F. Since M is not $\mathrm{GF}(q)$-representable, we have $r_M(F) \leq 2$. If $r_M(F) \geq m$, then $\mathrm{si}(M/(B-I)) \cong \widehat{\mathrm{PG}}(m-1,q;m)$, where I is an m-element subset of B_F. If $r_M(F) < m$, then $|B - B_F| \geq m-2$; it now follows that $\mathrm{si}(M/(J_1 \cup J_2)) \cong \widehat{\mathrm{PG}}(m-1,q;2)$, where $J_1 \subseteq B_F$ and $J_2 \subseteq B - B_F$ satisfy $|J_1| = r_M(F) - 2$ and $|J_2| = m - |J_1|$. □

We now restate and prove Theorem 1.1:

Theorem 2.6. *Let q be a prime power and let $m, \ell \geq 2$ be integers. If $M \in \mathcal{U}(\ell)$ is q-dense and $r(M)$ is sufficiently large, then M has a minor isomorphic to $\widehat{\mathrm{PG}}(m-1,q;2), \widehat{\mathrm{PG}}(m-1,q;m)$, or $\mathrm{PG}(m-1,q')$ for some $q' > q$.*

Proof. Recall that the function f was defined in Theorem 2.1. Let $n_1 = f(\ell, 2m+1, q, q^{-1})$ and let n_0 be an integer so that $(\sqrt{5}-1)^{n_0-1} \geq \ell^{n_1-1}$. We show that the conclusion holds whenever $r(M) \geq n_0$.

Let $M \in \mathcal{U}(\ell)$ be a q-dense matroid of rank at least n_0. By definition of n_0 and Lemma 2.4, M has a q-dense restriction M_1 such that $r(M_1) \geq n_1$ and every cocircuit of M_1 has rank at least $r(M_1) - 1$. Note that $\varepsilon(M_1) > q^{-1}q^{r(M_1)}$; by Theorem 2.1 and the definition of n_1, the matroid M_1 has an $\mathrm{AG}(2m,q)$-restriction R or a $\mathrm{PG}(2m,q')$-minor for some $q' > q$. In the latter case, the theorem holds. In the former case, let M_2 be a minimal minor of M_1 so that

(1) R is a restriction of M_2,
(2) every cocircuit of M_2 has rank at least $r(M_2) - 2$, and
(3) M_2 is either q-dense or has a $U_{2,q+2}$-restriction.

Note that $r(M_2) \geq r(R) \geq 5$, and that contracting any element not spanned by $E(R)$ gives a matroid satisfying (1) and (2). We argue that R is spanning in M_2; suppose not, and let $e \in E(M_2) - \mathrm{cl}_{M_2} E(R)$. If M_2 has a $U_{2,q+2}$-restriction $M_2|L$ containing e, then since $r(M_2) \geq 5$, the set $\mathrm{cl}_{M_2}(L)$ contains no cocircuit of M_2 and there is hence some $f \in E(M_2) - (\mathrm{cl}_{M_2}(E(R)) \cup \mathrm{cl}_{M_2}(L))$. Therefore $(M/f)|L \cong U_{2,q+2}$, contradicting minimality. Thus, M_2 has no $U_{2,q+2}$-restriction containing e, so Lemma 2.2 implies that M_2/e is q-dense, again contradicting minimality; therefore R is spanning in M_2.

If $f \in E(R)$, then M_2/f is a rank-$2m$ matroid with a $\mathrm{PG}(2m-1,q)$-restriction; it is thus enough to show that M_2/f is non-$\mathrm{GF}(q)$-representable for some such f, as the theorem then follows from Lemma 2.5. If M_2 has a $U_{2,q+2}$-restriction $M_2|L$, then any $f \in E(R) - L$

will do, since $(M_2/f)|L$ is not $\mathrm{GF}(q)$-representable. Otherwise, by Lemma 2.2, the matroid M_2/f is q-dense for any $f \in E(R)$; again this implies non-$\mathrm{GF}(q)$-representability. □

3. Lines, Spikes and Swirls

In this section, we restate and prove our four corollaries.

Theorem 3.1. *If $\ell \geq 2$ is an integer, then $h_{\mathcal{U}(\ell)}(n) = \frac{q^n-1}{q-1}$ for all sufficiently large n, where q is the largest prime power not exceeding ℓ.*

Proof. Note that $\mathcal{L}(q) \subseteq \mathcal{U}(\ell)$, giving $\frac{q^n-1}{q-1} \leq h_{\mathcal{U}(\ell)}(n) < \infty$ for all n. If the result fails, then by Theorem 1.3 we have either $U_{2,q^2+1} \in \mathcal{M}$ or $U_{2,q'+1} \in \mathcal{M}$, where q' is the smallest prime power such that $q' > \ell$. Clearly $q^2 \geq q' \geq \ell + 1$; it follows that $U_{2,\ell+2} \in \mathcal{U}(\ell)$, a contradiction. □

Our other corollaries depend on representability of free spikes and swirls. It can be easily shown that the free spike Λ_k is representable over a field $\mathrm{GF}(q)$ if and only if there exist nonzero $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2 \in \mathrm{GF}(q)$ so that $\beta_1 \neq \beta_2$ and no sub-multiset of the α_i sum to β_1 or β_2. The problem for Δ_k is analogous, but with products in the multiplicative group $\mathrm{GF}(q)^*$. Both problems are trivial unless the relevant group is of prime order, as one can choose the α_i in a subgroup not containing the β_i. Similarly, if the group has size at least $k+2$, one can choose the α_i all equal. The details for the prime-order case were dealt with in [2, Lemma 11.6]; the following lemma summarises the consequences:

Lemma 3.2. *If $k \geq 3$ is an integer and $q \geq 3$ is a prime power, then*
- *(1) $\Lambda_k \in \mathcal{L}(q)$ and only if q is composite or $k \leq q-2$.*
- *(2) $\Delta_k \in \mathcal{L}(q)$ if and only if $q-1$ is composite or $k \leq q-3$.*

It is easy to see that $\mathcal{L}^{\lambda}(q)$ contains every restriction of a matroid obtained from a matroid in $\mathcal{L}(q)$ by principally truncating a line. Moreover, $\mathcal{L}^{\circ}(q)$ contains all truncations of $\mathrm{GF}(q)$-representable matroids. We can now show that these classes contain all free spikes:

Lemma 3.3. *If q is a prime power and $k \geq 3$ is an integer, then $\Lambda_k \in \mathcal{L}^{\lambda}(q) \cap \mathcal{L}^{\circ}(q)$.*

Proof. Let $G \cong K_{2,k}$ and let $M = M(G)$. The free spike Λ_k is the truncation of the regular matroid M, so $\Lambda_k \in \mathcal{L}^{\circ}(q)$. Let H be a $K_{1,k}$-subgraph of G. For each prime power q, let $\widehat{M}$ be a $\mathrm{GF}(q)$-representable extension of M by a point e spanned by $E(H)$ but no proper subset of $E(H)$. Now we have $\Lambda_k \cong \widehat{M'} \backslash e$, where $\widehat{M'}$ is obtained from $\widehat{M}$ by principally truncating the line spanned by $\{e, f\}$ for some $f \in E(H)$. Therefore $\Lambda_k \in \mathcal{L}^\lambda(q)$. □

The same does not hold, however, for free swirls:

Lemma 3.4. *If $q \geq 3$ is a prime power and $k \geq 4$ is an integer, then*
- $\Delta_k \in \mathcal{L}^\lambda(q)$.
- $\Delta_k \in \mathcal{L}^\circ(q)$ *if and only if* $\Delta_k \in \mathcal{L}(q)$.

Proof. Let $L_1, L_2, \ldots, L_k$ be copies of $U_{2,4}$ so that $|E(L_i) \cap E(L_{i+1})| = 1$ for each $i \in \{1, \ldots, k-1\}$ and $E(L_i) \cap E(L_j) = \varnothing$ for $|i - j| > 1$. Let $x_1 \in E(L_1) - E(L_2)$ and $x_k \in E(L_k) - E(L_{k-1})$. Let N_k be defined by the repeated 2-sum $L_1 \oplus_2 L_2 \oplus_2 \ldots \oplus_2 L_k$. Clearly $N_k \in \mathcal{L}(q)$, and $\widehat{N_k} \setminus \{x_1, x_k\} \cong \Delta_k$, where $\widehat{N_k}$ is the principal truncation of the line spanned by x_1 and x_k in N_k. Therefore $\Delta_k \in \mathcal{L}^\lambda(q)$.

On the other hand, suppose that Δ_k is in exactly one of $\mathcal{L}(q)$ and $\mathcal{L}^\circ(q)$. Since $\mathcal{L}(q) \subseteq \mathcal{L}^\circ(q)$, it must be the case that Δ_k is the truncation of a rank-$(k+1)$ matroid $N \in \mathcal{L}(q)$. Let $E(\Delta_k) = P_1 \cup \ldots \cup P_k$, where the P_i are pairwise disjoint two-element sets so that the union of any two cyclically consecutive P_i is a circuit of Δ_k, and the union of two any other P_i is independent in Δ_k. Since $r(N) \geq 5$ and Δ_k is the truncation of N, we thus have $N|(P_i \cup P_j) = \Delta_k|(P_i \cup P_j)$ for all distinct i and j. As $P_i \cup P_{i+1}$ is a circuit of N for each $i < k$, an inductive argument gives $r_N(P_1 \cup \ldots \cup P_{k-1}) \leq k$. However $r_N(P_{k-1} \cup P_1 \cup P_k) \leq 4$ by submodularity, so $P_k \subset \mathrm{cl}_N(P_{k-1} \cup P_1)$ and $r(N) \leq k$, a contradiction. □

The fact that $\mathcal{L}^\circ(q)$ need not contain all free swirls is the reason that Theorem 1.7 is more technical and less powerful than Theorem 1.6. We now restate and prove both these theorems:

Theorem 3.5. *Let $k \geq 3$ and $\ell \geq 2$ be integers. If $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$- or Λ_k-minor, then $h_{\mathcal{M}}(n) = \frac{p^n-1}{p-1}$ for all sufficiently large n, where p is the largest prime satisfying $p \leq \min(\ell, k+1)$.*

Proof. By Lemma 3.2, we have $\Lambda_k \notin \mathcal{L}(p)$ and so $\mathcal{L}(p) \subseteq \mathcal{M}$ and $\frac{p^n-1}{p-1} \leq h_{\mathcal{M}}(n) < \infty$ for all n. If the result does not hold, then by Theorem 1.3 the class $\mathcal{M}$ contains $\mathcal{L}^\circ(p), \mathcal{L}^\lambda(p)$ or $\mathcal{L}(q)$ for some prime power $q > p$. In the first two cases we have $\Lambda_k \in \mathcal{M}$, a contradiction. In the last case, since $U_{2,\ell+2} \notin \mathcal{L}(q)$ and $\Lambda_k \notin \mathcal{L}(q)$, we know by Lemma 3.2 that q is prime and $q \leq \max(\ell, k+1)$; this contradicts the maximality in our choice of p. □

Theorem 3.6. *Let $2^p - 1$ and $2^{p'} - 1$ be consecutive Mersenne primes, and let k and ℓ be integers for which $2^p \leq \ell < \min(2^{2p} + 2^p, 2^{p'})$ and*

$k \geq \max(4, 2^p - 2)$. *If* $\mathcal{M}$ *is the class of matroids with no* $U_{2,\ell+2}$*- or* Δ_k*-minor, then* $h_{\mathcal{M}}(n) = \frac{2^{pn}-1}{2^p-1}$ *for all sufficiently large* n.

Proof. Since $\ell \geq 2^p$ and $k \geq 2^p - 2$, we have $U_{2,\ell+2} \notin \mathcal{L}(2^p)$ and $\Lambda_k \notin \mathcal{L}(2^p)$, so $\mathcal{L}(2^p) \subseteq \mathcal{M}$, giving $2^{np} - 1 \leq h_{\mathcal{M}}(n) < \infty$ for all n. If the result fails, then $\mathcal{M}$ contains $\mathcal{L}^\circ(2^p), \mathcal{L}^\lambda(2^p)$ or $\mathcal{L}(q)$ for some prime power $q > 2^p$. We have $U_{2,2^{2p}+2^p+1} \in \mathcal{L}^\circ(2^p)$, and $\Delta_k \in \mathcal{L}^\lambda(2^p)$ by Lemma 3.4. If $q-1$ is composite, then $\Delta_k \in \mathcal{L}(q)$. If $q-1$ is prime, then $q \geq 2^{p'}$, so $U_{2,2^{p'}+1} \in \mathcal{L}(q)$. Since $\ell < \min(2^{2p} + 2^p, 2^{p'})$, we have $U_{2,\ell+2} \in \mathcal{M}$ or $\Lambda_k \in \mathcal{M}$ in all cases, a contradiction. □

We cannot hope for such a simple theorem applying to all ℓ; to see why, suppose that $p' > 2p$ (for example, if $(p, p') = (127, 521)$). Then if $2^{2p} + 2^p \leq \ell < 2^{p'}$ and $k \geq 2^p - 2$, it follows from Lemmas 3.2 and 3.4 that $\mathcal{L}^\circ(2^p) \subseteq \mathcal{M}$ but $\mathcal{L}(q) \nsubseteq \mathcal{M}$ for all $q > 2^p$. The Growth rate theorem thus gives $\frac{2^{p(n+1)}-1}{2^p-1} \leq h_{\mathcal{M}}(n) \leq c \cdot 2^{pn}$ for some constant c, so $h_{\mathcal{M}}(n)$ does not eventually equal $\frac{q^n-1}{q-1}$ for any prime power q.

Finally, we prove Theorem 1.8:

Theorem 3.7. *Let* $\ell \geq 3$ *and* $k \geq 3$ *be integers. If* $\mathcal{M}$ *is the class of matroids with no* $U_{2,\ell+2}$*-,* Λ_k*- or* Δ_k*-minor, then* $h_{\mathcal{M}}(n) = \frac{1}{2}(3^n - 1)$ *for all sufficiently large* n.

Proof. As before, if the theorem fails, $\mathcal{M}$ contains $\mathcal{L}^\lambda(3), \mathcal{L}^\circ(3)$ or $\mathcal{L}(q)$ for some $q > 3$. In the first two cases, we have $\Lambda_k \in \mathcal{M}$, and otherwise, since either q or $q-1$ is composite, we have $\Lambda_k \in \mathcal{M}$ or $\Delta_k \in \mathcal{M}$, a contradiction. □

Acknowledgements

I thank Jim Geelen for bringing the corollaries regarding spikes and swirls to my attention.

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