A first step in decomposing near-regular matroids

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(Joint work with D. Slilaty and D. Chun)

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Theorem (Tutte). If $GF(2) \in \mathcal{F}$, then $\mathcal{M}(\mathcal{F})$ is the class of either

1) regular matroids, or
2) binary matroids.

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Theorem (Whittle). If $GF(3) \in \mathcal{F}$, then $\mathcal{M}(\mathcal{F})$ is the class of either

1) regular matroids, $GF(2) \in \mathcal{F}$
2)
3)
4)
5)
6) ternary matroids.

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3) $\sqrt[6]{1}$-matrois $GF(4) \in \mathcal{F}$
4) dyadic matroids, $GF(5) \in \mathcal{F}$
5) sums of 3) & 4), $GF(7) \in \mathcal{F}$
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Theorem (Whittle). If $GF(3) \in \mathcal{F}$, then $\mathcal{M}(\mathcal{F})$ is the class of either

1) regular matroids, $GF(2) \in \mathcal{F}$
2) near-regular matroids, $GF(4), GF(5) \in \mathcal{F}$
3) $\sqrt[6]{1}$-matrois $GF(4) \in \mathcal{F}$
4) dyadic matroids, $GF(5) \in \mathcal{F}$
5) sums of 3) & 4), $GF(7) \in \mathcal{F}$
6) ternary matroids.

Regular matroids are well understood:

- A matroid is regular iff it is representable over every field.

- Excluded minors

- Theorem (Seymour). Every regular matroid may be constructed from graphic and cographic matroids and copies of R_{10} via 1-, 2- and 3-sums.

- Recognition algorithm

Near-regular matroids are only partially understood:

- A matroid is near-regular iff it is representable over every field except possibly GF(2).

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  Theorem (Hall, Mayhew and van Zwam). The excluded minors are $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, F_7^-, $(F_7^-)^*$, P_8, $AG(2,3)\backslash e$, $(AG(2,3)\backslash e)^*$ and $\Delta_T(AG(2,3)\backslash e)$.

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- Decomposition theorem ??

- Recognition algorithm ??

Conjecture (Mayhew, van Zwam and Whittle)

Every near-regular matroid can be obtained from signed-graphic matroids, their duals and matroids from a finite list through 1-, 2- and 3-sums and k-clique sums (or their duals) for $k \leq 4$.

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Not all signed-graphic matroids are near-regular.

Slilaty and Qin characterised the near-regular signed-graphic matroids using results by Pagano.

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Theorem (Chun, P., Slilaty)

The conjecture holds for single-element extensions and co-extensions of graphic matroids.

Some bits of the proof ...

We work over $GF(4) = \{0, 1, w, w^2\}$

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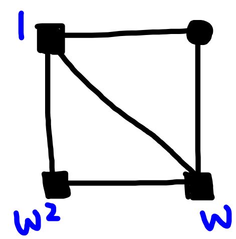
Extensions of graphic matroids

$$\begin{array}{cc} \overbrace{\hphantom{1\quad 1\quad 1\quad 0\quad 0}}^{M(G)} & \vec{T} \\ \multicolumn{2}{c}{\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & \omega \\ 0 & 0 & 1 & 0 & 1 & \omega^2 \end{array}\right]} \end{array} \longrightarrow$$

GF(4)-graft



■ = T-vertices

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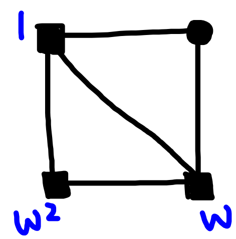
Extensions of graphic matroids

$$
\begin{array}{c}
\begin{array}{cc} \overbrace{\hspace{9em}}^{M(G)} & \vec{T} \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & w \\
0 & 0 & 1 & 0 & 1 & w^2
\end{array}\right]
\end{array}
\longrightarrow
$$

GF(4)-graft

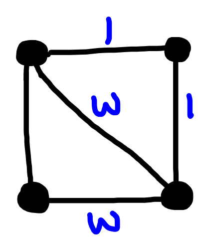


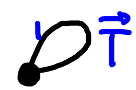
■ = T-vertices

Coextensions of graphic matroids

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & & & & & \vec{T} \end{array} \\
\begin{array}{c} \\ M(G) \\ \\ \\ T \end{array} &
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & w & 0 & 1 & w & 1
\end{array}\right]
\end{array}
\longrightarrow
$$

GF(4) - gain graph





Some bits of the proof ...

We make the graph basically 3-connected.

We use 1-, 2- and 3-sums, plus :

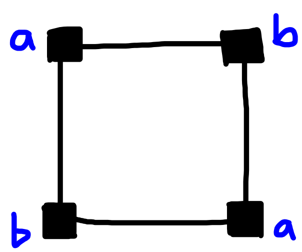
- for extensions we need to include dual 3-sums.
- for coextensions we need 4-clique sums.

Some bits of the proof ...

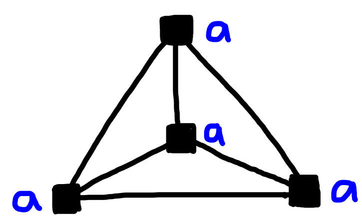
Which of the excluded-minors for near-regular matroids can be represented as GF(4)-grafts?

- F_7^-, $(F_7^-)^*$ and P_8 are not GF(4)-rep.
- $U_{2,5}$ and the AG(2,3) family are not extensions of graphic matroids.

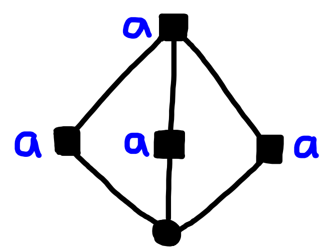
$U_{3,5}$



F_7



F_7^*

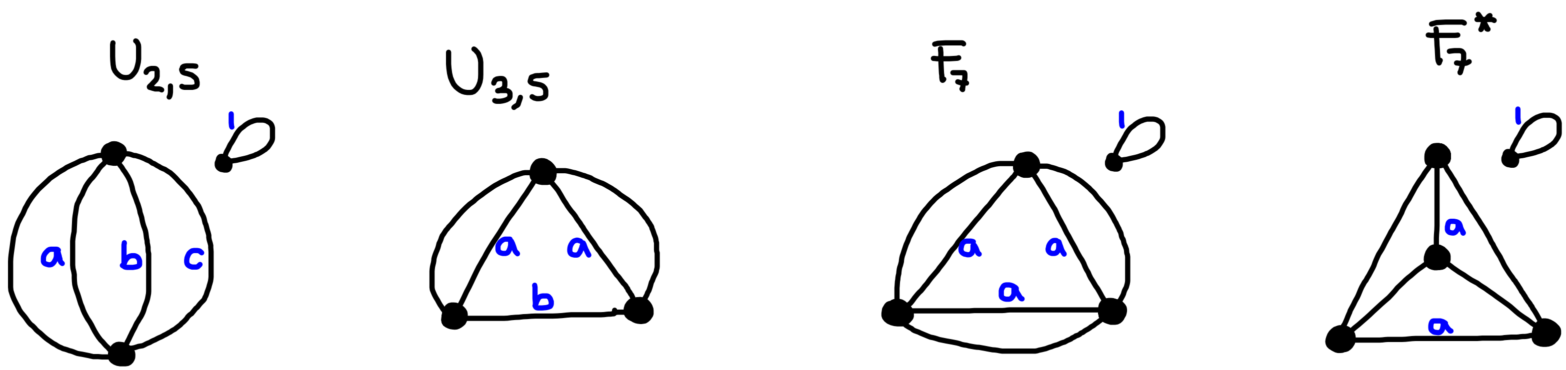


$a, b \in GF(4)\setminus\{0\}$, $a \neq b$

Some bits of the proof ...

Which of the excluded-minors for near-regular matroids can be represented as GF(4)-gain graphs?

- F_7^-, $(F_7^-)^*$ and P_8 are not GF(4)-rep.
- matroids in the AG(2,3) family are not coextensions of graphic matroids.



$a, b, c \in GF(4) \setminus \{0\}$, a, b, c all distinct

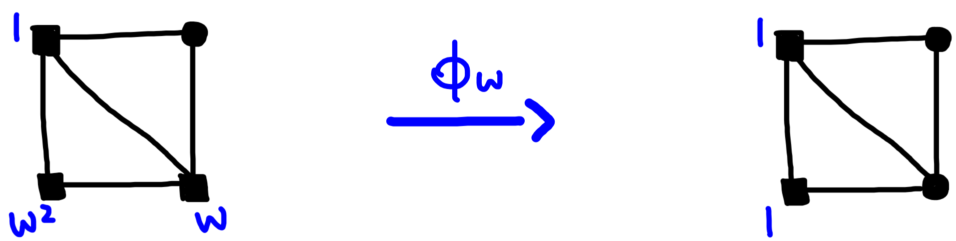
Some bits of the proof ...

We use quotients:

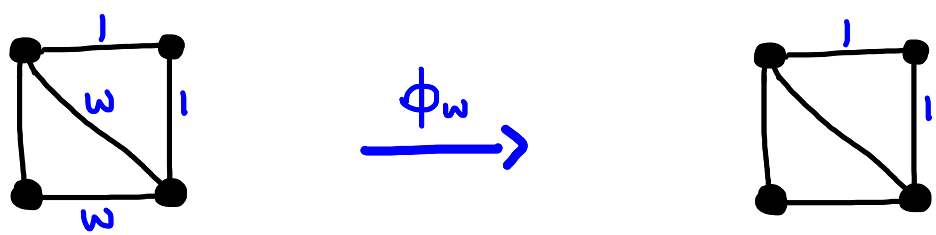
For $a \in GF(4)\setminus 0$, define $\phi_a : GF(4)^+ \longrightarrow GF(4)^+/\langle a \rangle \simeq GF(2)$

E.g. $\phi_\omega(0) = \phi_\omega(\omega) = 0$ and $\phi_\omega(1) = \phi_\omega(\omega^2) = 1$.

GF(4)-graft　　　　　　　　graft

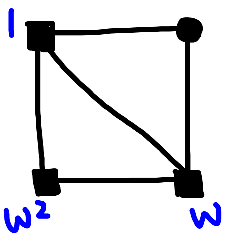


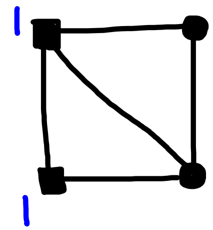
GF(4)-gain graph　　　　　　　　Signed graph



Some bits of the proof ...

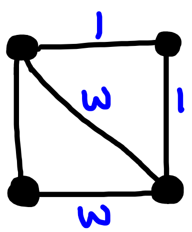
Lemma:

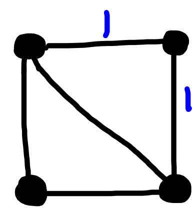
1) $M\Big($ 

$\Big) \xrightarrow{\phi_w} M\Big($ 

$\Big)$

is near regular $\Rightarrow$ is regular

2) $M\Big($ 

$\Big) \xrightarrow{\phi_w} M\Big($ 

$\Big)$

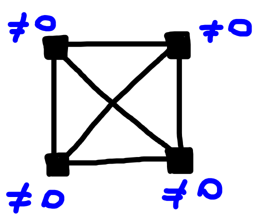
is near regular $\Rightarrow$ is regular

Some bits of the proof ...

For GF(4)-grafts :

Seymour used grafts to prove the decomposition theorem for regular matroids and we use one of his results.

But we need quite a bit more work ...

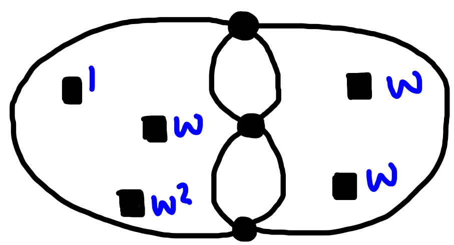
If we have 

as a minor, then the matroid is not near regular.

We use results by Fabila and Wood on rooted K_4-minors.

Some bits of the proof ...

Problem: taking the quotient may create 3-separations with a graphic side.

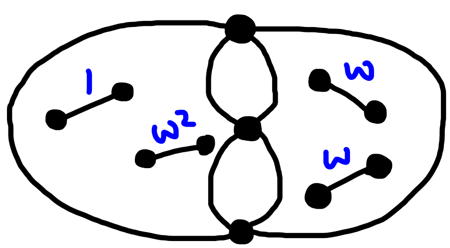


Some bits of the proof ...

For GF(4)-gain graphs

Gerards characterised the signed graphs that give regular matroids.

Problem: taking the quotient may create 3-separations with a graphic side.




THANK YOU!