MERINO-WELSH CONJECTURE FOR SERIES-PARALLEL GRAPHS

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OUTLINE

1 THE CONJECTURE(S)

2 SERIES-PARALLEL GRAPHS

THE PARTICIPANTS I

Criel Merino

Dominic Welsh

THE PARTICIPANTS II

Steve Noble

Activities & Hobbies

- The Tutte Polynomial
- Head of Department[a] at Brunel
- CEROC
- 1930s jazz
- Wolverhampton Wanderers
- Amusing impersonations of famous mathematicians, especially after a beer or two.

[a] Subject to change

THE CONJECTURE(S)

Let G be a graph, and denote by:

- $\tau(G)$, the number of *spanning trees* of G,
- $\alpha(G)$, the number of *acyclic orientations* of G,
- $\alpha^*(G)$, the number of *totally cyclic orientations* of G.

The *Merino-Welsh conjecture* – *original* version – is that if G is *bridgeless* and *loopless*, then:

$$\tau(G) \leq \max\{\alpha(G), \alpha^*(G)\}.$$

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The *Merino-Welsh conjecture* – *additive* – is that if G is *bridgeless* and *loopless*, then:

$$2\tau(G) \leq \alpha(G) + \alpha^*(G).$$

THE CONJECTURE(S)

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- $\tau(G)$, the number of *spanning trees* of G,
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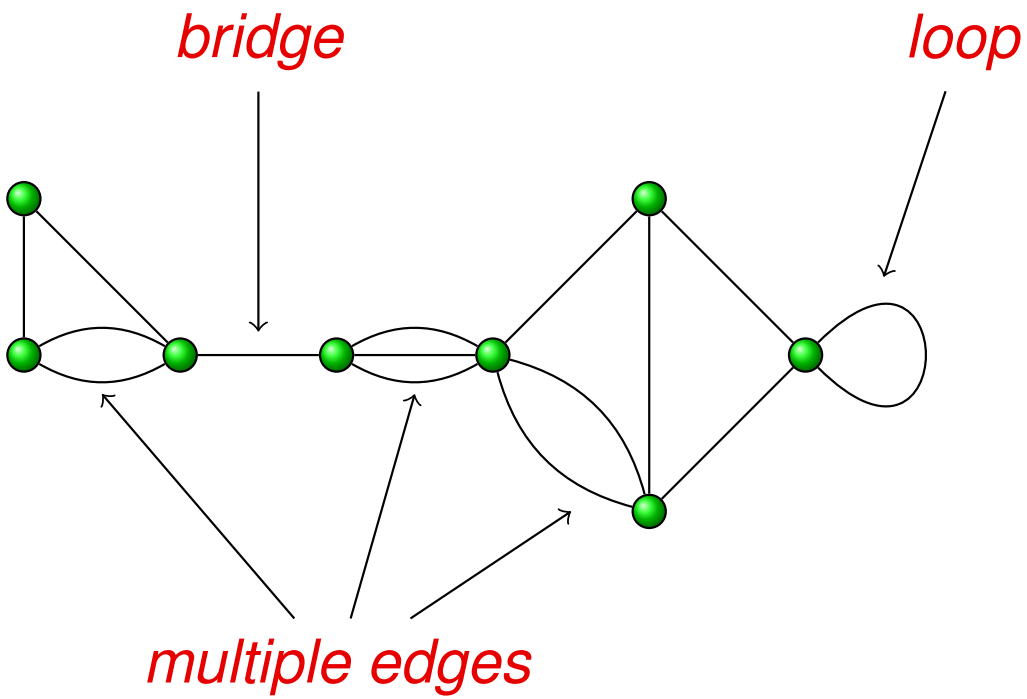
The *Merino-Welsh conjecture* – *multiplicative* version – is that if G is *bridgeless* and *loopless*, then:

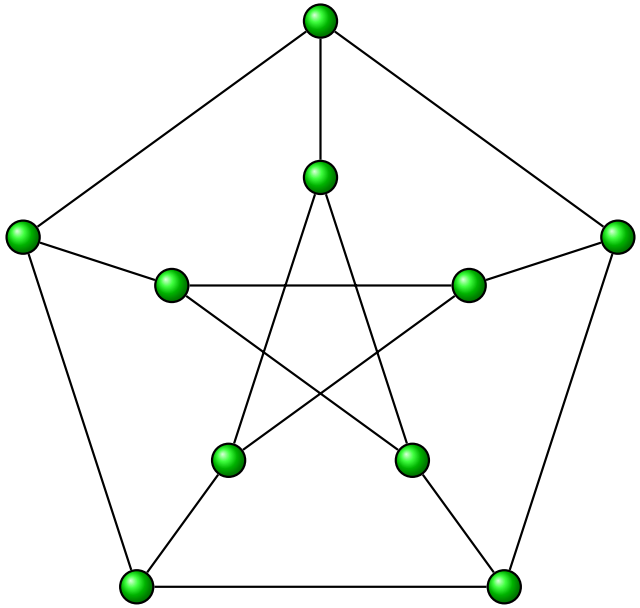
$$\tau(G)^2 \leq \alpha(G) \cdot \alpha^*(G).$$

GRAPHS, BRIDGES AND LOOPS



THE THREE PARAMETERS

We all know what *spanning trees* are:



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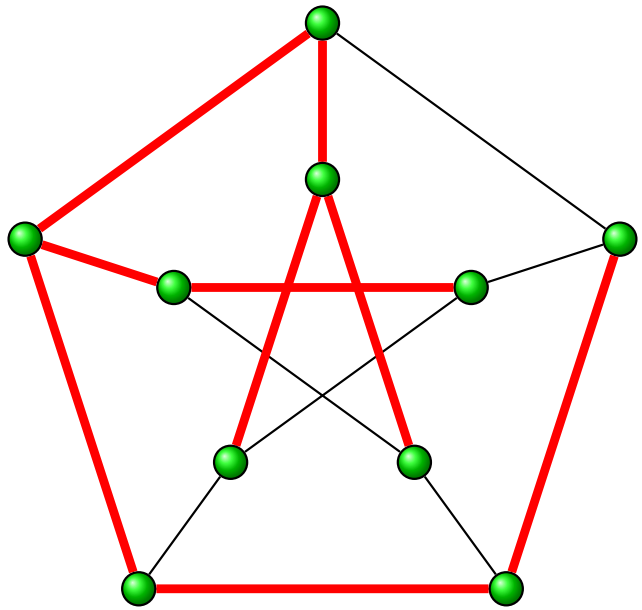
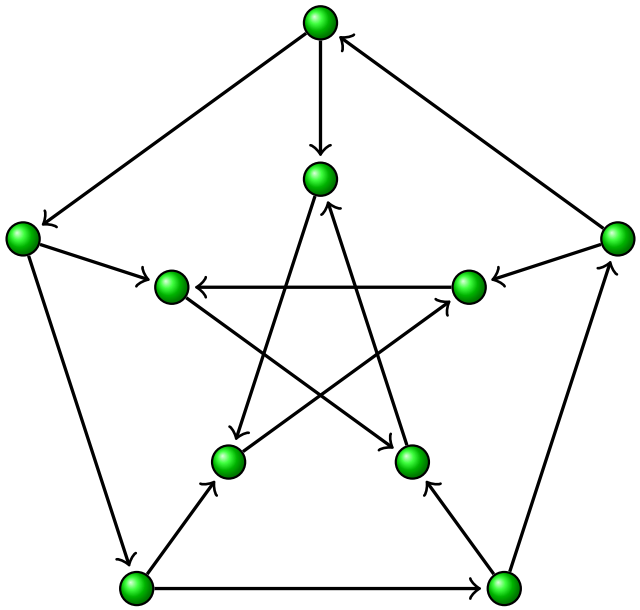


The *Petersen* graph P has 2000 spanning trees, so $\tau(P) = 2000$.

WHAT ARE ORIENTATIONS?

An *orientation* of a graph is the assignment of a *direction* to each edge:



WHAT ABOUT ACYCLIC AND TOTALLY CYCLIC?

Let D denote the directed graph determined by an orientation. Then the orientation is

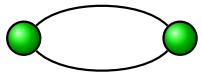
- *acyclic* if D has *no directed cycles*,
- *totally cyclic* if D has a directed cycle through *every edge*.

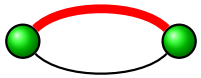
The Petersen graph has

- $\tau(P) = 2000$ spanning trees,
- $\alpha^*(P) = 1920$ totally cyclic orientations,
- $\alpha(P) = 16680$ acyclic orientations.

All three versions of the Merino-Welsh conjecture hold.

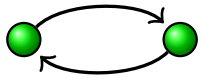
EQUALITY

The graph  has *two* spanning trees,



two acyclic orientations

and *two* totally cyclic orientations:

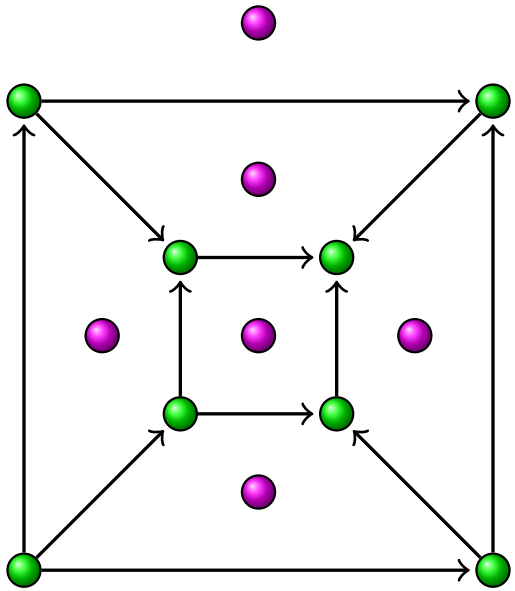


and so

$$\tau = \alpha = \alpha^* = 2.$$

DUALITY

If G is a *plane graph* then an orientation of G determines an orientation of G^*:



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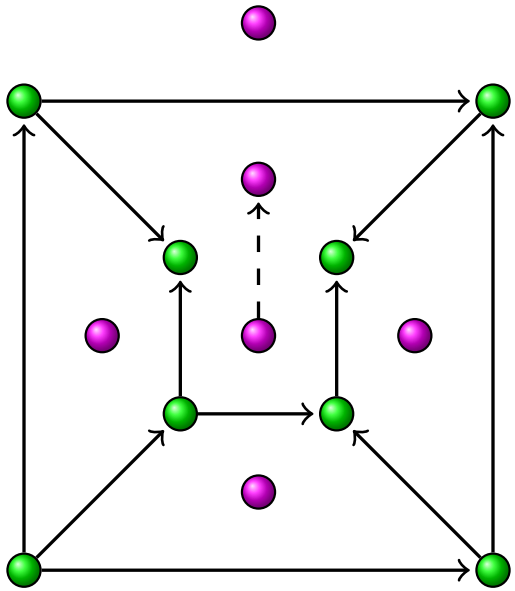

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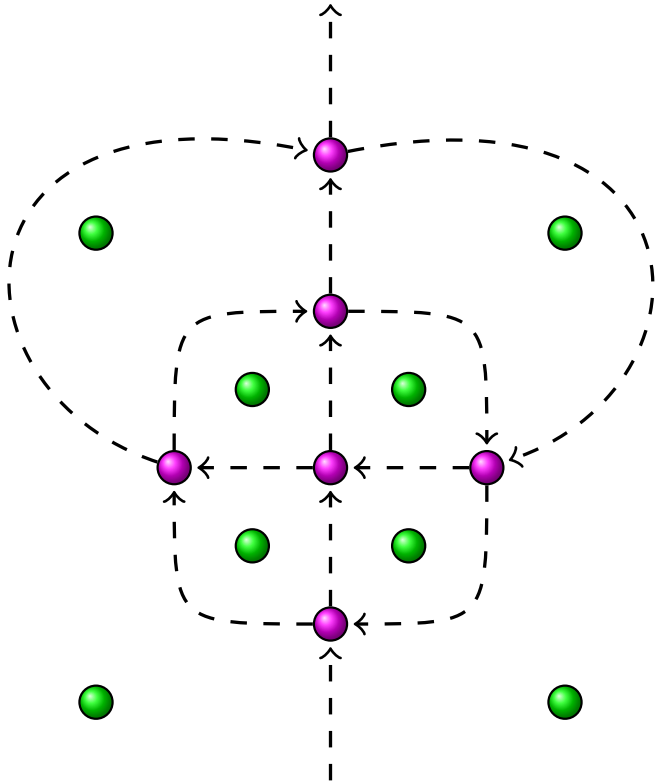


DUALITY II

An edge e *does* lie in a directed cycle in G if and only if it *does not* lie in a directed cycle in G^*, and so an *acyclic orientation* of G is a *totally cyclic orientation* of G^*. Therefore

$$\tau(G^*) = \tau(G)$$
$$\alpha(G^*) = \alpha^*(G)$$
$$\alpha^*(G^*) = \alpha(G)$$

Therefore, all three version of the Merino-Welsh conjecture are *closed under duality*.

WHAT IF THE GRAPH IS NOT PLANAR?

All three parameters are evaluations of the *Tutte polynomial*:

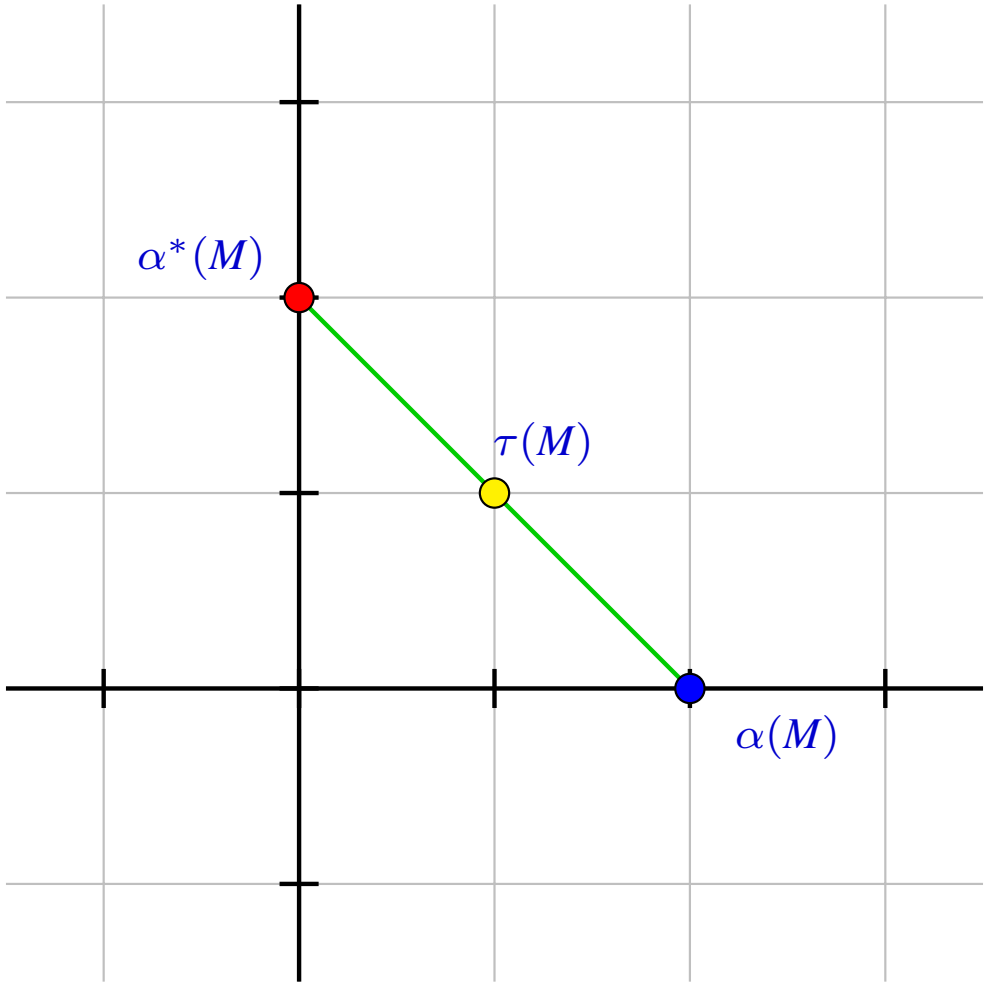
$$\tau(G) = T_G(1,1),$$
$$\alpha(G) = T_G(2,0),$$
$$\alpha^*(G) = T_G(0,2).$$

The standard technique for extending a *graph* conjecture to *matroids* [1] yields the general Merino-Welsh conjecture: for any bridgeless loopless matroid M,

$$T_M(1,1) \leq \max\{T_M(2,0), T_M(0,2).\}$$

[1] i.e replacing G by M

THE TUTTE PLANE

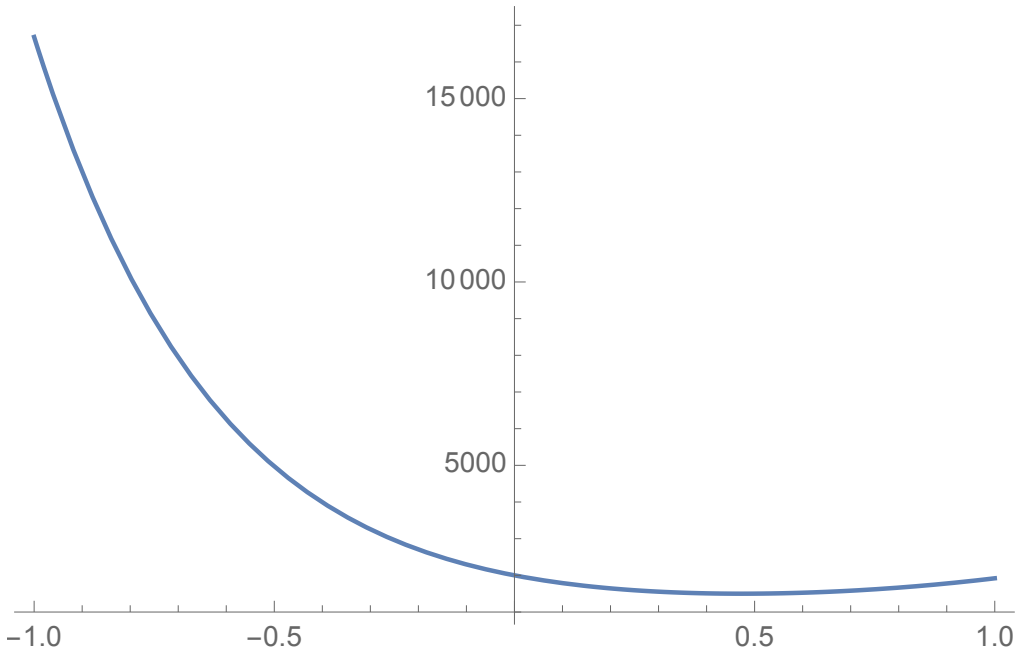


CONTINUOUS VERSION

For any matroid M, the function

$$\varphi_M(x) = T_M(1-x, 1+x),$$

is *convex* in the region $-1 \leq x \leq 1$.



PAVING MATROIDS

Chávez-Lomelí, Merino, Noble and Ramírez-Ibáñez proved this conjecture for all *paving matroids* (matroids with no circuits of size less than rank).

Paving matroids form a huge class of matroids — so huge that it is conjectured that *almost all matroids* are paving.

But almost no graphs are paving matroids.

SPARSE GRAPHS

If a graph is *sparse* then it has lots of *acyclic* orientations:

THEOREM *(Thomassen) If G is a simple graph on n vertices with $m \leq 16n/15$ edges, then $\tau(G) < \alpha(G)$.*

If a graph is *dense* then it has lots of *totally cyclic* orientations:

THEOREM *(Thomassen) If G is a bridgeless graph on n vertices with $m \geq 4n - 4$ edges, then $\tau(G) < \alpha^*(G)$.*

A POSSIBLE APPROACH

QUESTION

Thomassen asked whether it could possibly be the case that if G is a graph with n vertices and m edges, that:

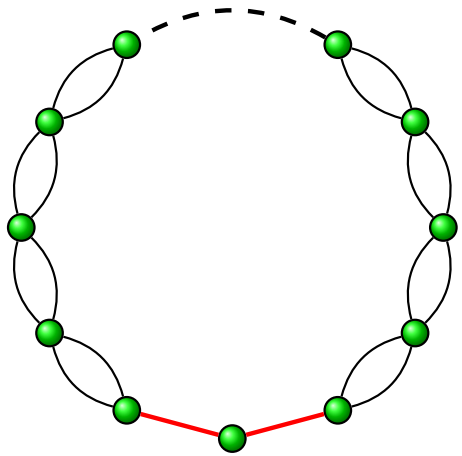
- $\tau(G) \leq \alpha(G)$ if $m \leq 2n-2$, and
- $\tau(G) \leq \alpha^*(G)$ if $m \geq 2n-2$.

If $m(G) \leq 2(n(G)-1)$, then $m(G^*) \geq 2(n(G^*)-1)$ and so, providing we are cavalier with duality[2], these are equivalent.

[2] or willing to move to cographic matroids.

BUT THEY'RE NOT TRUE

Suppose G has $n-2$ digons and 2 edges in a cycle; then $m(G) = 2n(G) - 2$.



$$\tau(G) = 2^{n-1} + (n-2)2^{n-3}$$
$$\alpha(G) = 2^n - 2$$
$$\alpha^*(G) = 2 \cdot 3^{n-2}$$

If $n \geq 6$ then

$$\alpha(G) < \tau(G) < \alpha^*(G).$$

OUTLINE

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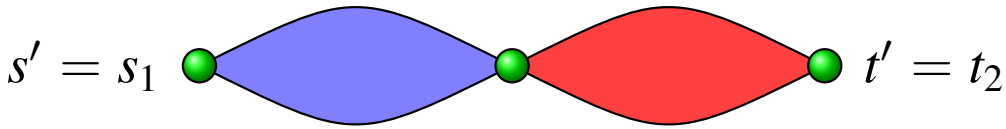
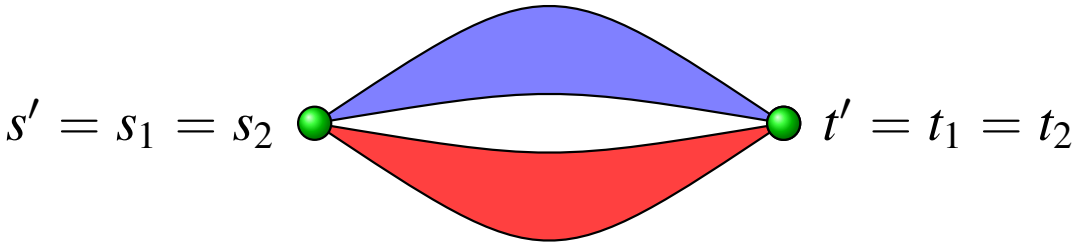
SERIES AND PARALLEL CONNECTION

A *2-terminal graph* (G, s, t) is a graph G with two distinguished vertices s and t (the *terminals*).

If (G_1, s_1, t_1) and (G_2, s_2, t_2) are 2-terminal graphs then their *series connection* $G_1 \oplus_S G_2$ is:

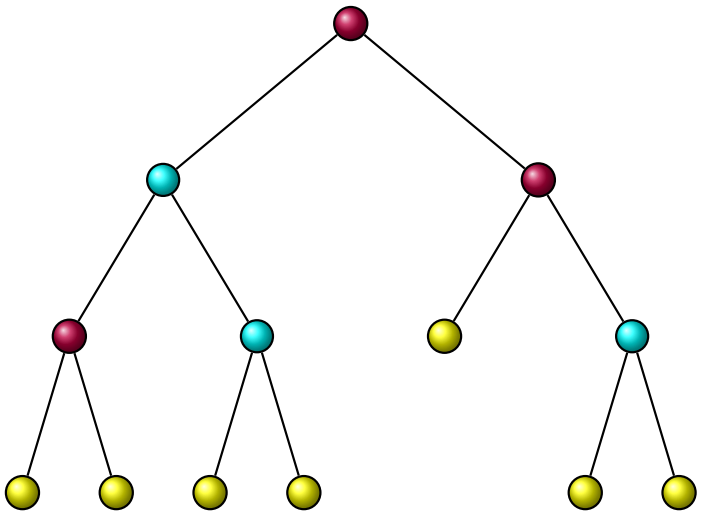


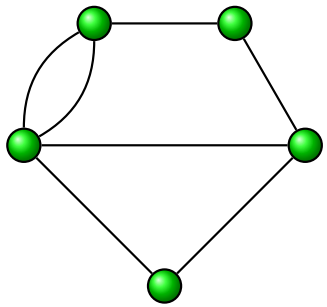
and their *parallel connection* $G_1 \oplus_P G_2$ is:



SERIES-PARALLEL GRAPHS

A graph is *2-terminal series-parallel* if it can be built from K_2 by repeated series and parallel connection, and *series-parallel* if it is 2-terminal series-parallel for *some* pair of vertices.[3]





[3] Many other characterisations of series-parallel graphs are known, e.g partial 2-trees, K_4-minor-free etc.

MAIN RESULT

Our main result is that the *multiplicative* version of the Merino-Welsh conjecture is true for series-parallel graphs.

THEOREM If G is a loopless bridgeless series-parallel graph, then

$$\tau(G)^2 \leq \alpha(G)\alpha^*(G).$$

Although only a tiny subset of graphs, they are centred about the "*critical point*" $m = 2n - 2$.

In addition, our proof is *not based* on showing that either α or α^* dominates.

PROOF FLAVOUR

We aim to show the non-existence of a putative minimal counterexample in the following way:

- Note that a minimal counterexample is *2-connected*.
- Define a concept of *reducibility* on 2-terminal s/p-graphs.
- Classify the *irreducible* graphs and show they all satisfy the conjecture.

KEEPING COUNT

The natural recursive structure makes *"counting type"* problems easier — but only if you know what to count. We need to keep some "auxiliary" data:

- A 2-*forest* of (G, s, t) is a set of edges that induces a 2-component forest with s and t in different components.
- A *very acyclic* orientation of (G, s, t) is an acyclic orientation with no directed path between s and t.
- An *almost totally cyclic* orientation of (G, s, t) is an orientation in which every edge is either in a directed cycle or a directed path between s and t.

Use $\tau_2(G)$, $\alpha_2(G)$ and $\alpha_2^*(G)$ to denote their respective numbers.

PARALLEL CONNECTION

If $G = G_1 \oplus_P G_2$ then

$$\begin{aligned}
\tau(G) &= \tau(G_1)\tau_2(G_2) + \tau_2(G_1)\tau(G_2),\\
\tau_2(G) &= \tau_2(G_1)\tau_2(G_2),\\
\alpha(G) &= \alpha(G_1)\alpha(G_2) - \frac{(\alpha(G_1) - \alpha_2(G_1))(\alpha(G_2) - \alpha_2(G_2))}{2},\\
\alpha_2(G) &= \alpha_2(G_1)\alpha_2(G_2),\\
\alpha_2^*(G) &= \alpha_2^*(G_1)\alpha_2^*(G_2),\\
\alpha^*(G) &= \alpha_2^*(G_1)\alpha_2^*(G_2) - \frac{(\alpha_2^*(G_1) - \alpha^*(G_1))(\alpha_2^*(G_2) - \alpha^*(G_2))}{2}.
\end{aligned}$$

(A *dual statement* holds for series connection.)

REDUCIBILITY

A graph[4] G is *reducible* if a smaller graph H if for all graphs K:

$$\text{if } H \oplus_P K \text{ is ok , then } G \oplus_P K \text{ is ok,}$$

where "ok" is shorthand for

"satisfies the multiplicative Merino-Welsh conjecture."

A minimal counterexample cannot "use" G, because we could replace it by H instead.

[4] that is, a 2-terminal series-parallel graph

TESTING REPLACEABILITY

To *test* replaceability, note that

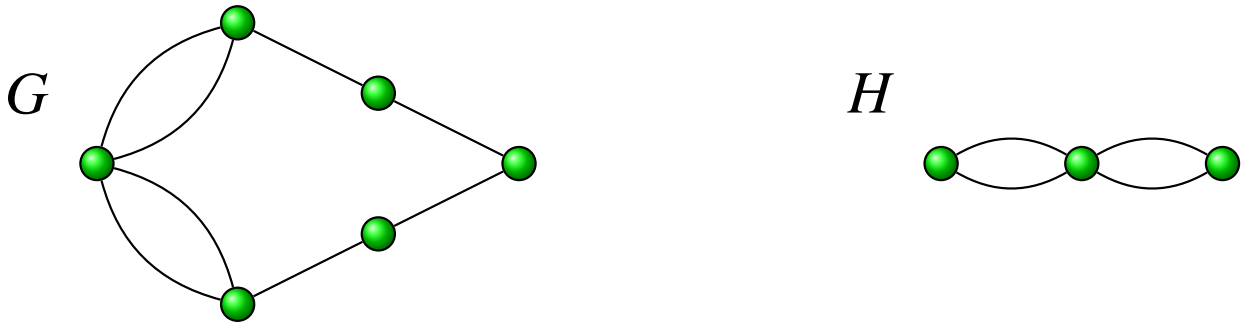
$$\tau(G \oplus_P K) = t_1\tau(G) + t_2\tau_2(G)$$
$$\alpha(G \oplus_P K) = a_1\alpha(G) + a_2\alpha_2(G)$$
$$\alpha^*(G \oplus_P K) = c_1\alpha^*(G) + c_2\alpha_2^*(G)$$

where t_1, t_2, a_1, a_2, c_1, c_2 depend only on K.

More precisely, $t_1 = \tau(K)$, $t_2 = \tau_2(K)$, and so on.

REDUCIBILITY

How can we show that a graph is reducible?



$$(\tau(G), \tau_2(G), \alpha(G), \alpha_2(G), \alpha_2^*(G), \alpha_2(G)) = (20, 25, 62, 36, 36, 18)$$
$$(\tau(H), \tau_2(H), \alpha(H), \alpha_2(H), \alpha_2^*(H), \alpha_2(H)) = (4, 4, 4, 2, 14, 4)$$

Claim: G is *reducible*, with H witnessing this.

REDUCIBILITY

If $H \oplus_P K$ is ok, then for t_1, t_2, a_1, a_2, c_1, c_2 determined by K

$$(4t_1 + 4t_2)^2 \leq (4a_1 + 2a_2) \cdot (14c_1 + 4c_2).$$

Replacing H with G, we have

$$\begin{aligned}(4t_1 + 4t_2)^2 &\to (20t_1 + 25t_2)^2\\ (4a_1 + 2a_2) &\to (62a_1 + 36a_2)\\ (14c_1 + 4c_2) &\to (36c_1 + 18c_2)\end{aligned}$$

Thus the left-hand side is multiplied by *at most* $(25/4)^2$ and the right-hand side by *at least* $(62/4) \cdot (36/14)$, and so $G \oplus_P K$ is ok.

AN ALGORITHM

We try to find a minimal counterexample by *systematically* building 2-terminal s/p-graphs starting with K_2, and then by increasing numbers of edges.

At every stage, choose two graphs G and H from the list and

- form $G \oplus_S H$ and $G \oplus_P H$,
- add them to the list unless they are *reducible* (as witnessed by smaller graphs already in the list).
- avoid creating any series or parallel classes of size more than 3 (ignore this for now).

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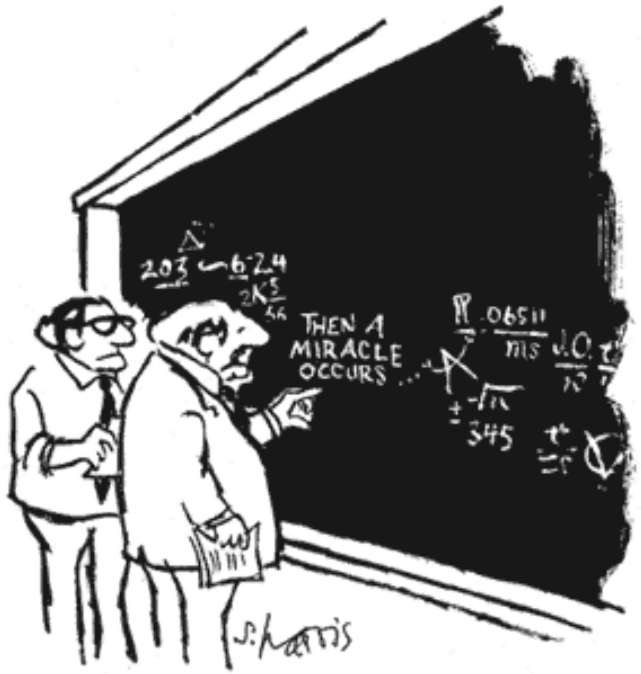
Every graph not on this list is *reducible*.

AND THEN . . .



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

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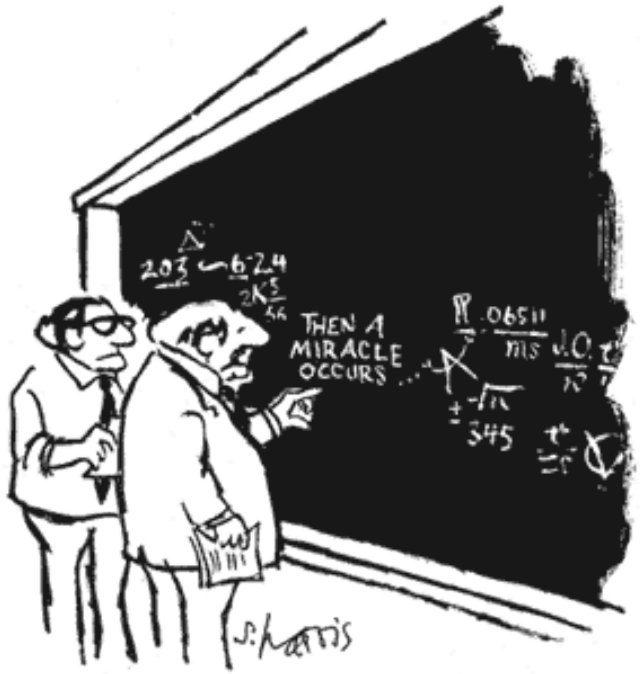

"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

. . . the process terminates!

THE DÉNOUEMENT

Let G be a minimal counterexample

- G is 2-connected
- $G = G_1 \oplus_P G_2$
- G_1 and G_2 are irreducible
- G_1 and G_2 are on the list

But, the list only contains a small number of tiny graphs, all of which satisfy the conjecture.

CONCLUSION

- Conjecture true for series-parallel graphs,
- Proof has ~~*no chance of generalising*~~ already been generalised!
- New ideas needed, especially for $m = 2n - 2$,
- Are there graphs with $m < 2n - 2$ edges with $\tau(G) > \alpha(G)$?