A splitter theorem for internally 4-connected graphs and binary matroids

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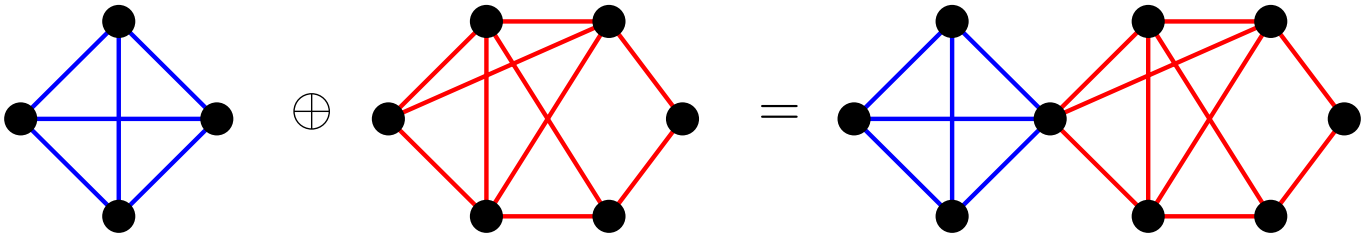
Nous sommes les moutons. July, 2015

Connectivity



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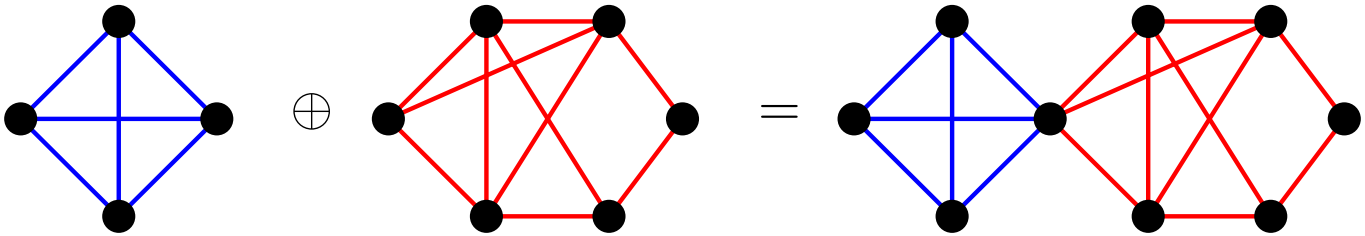

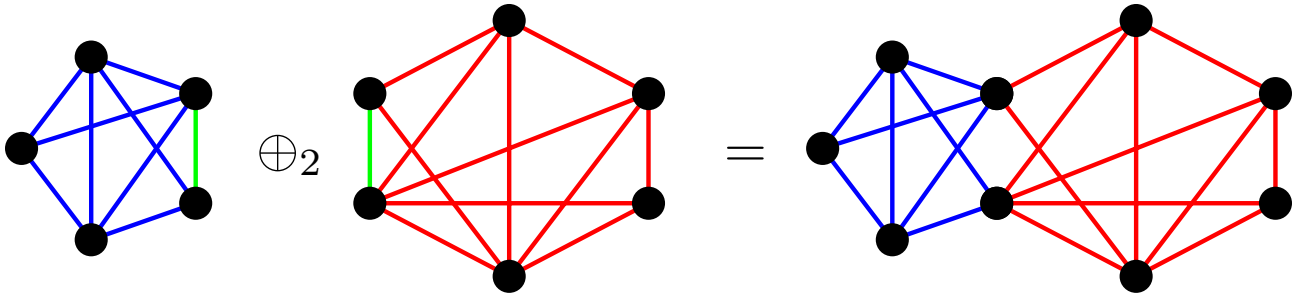

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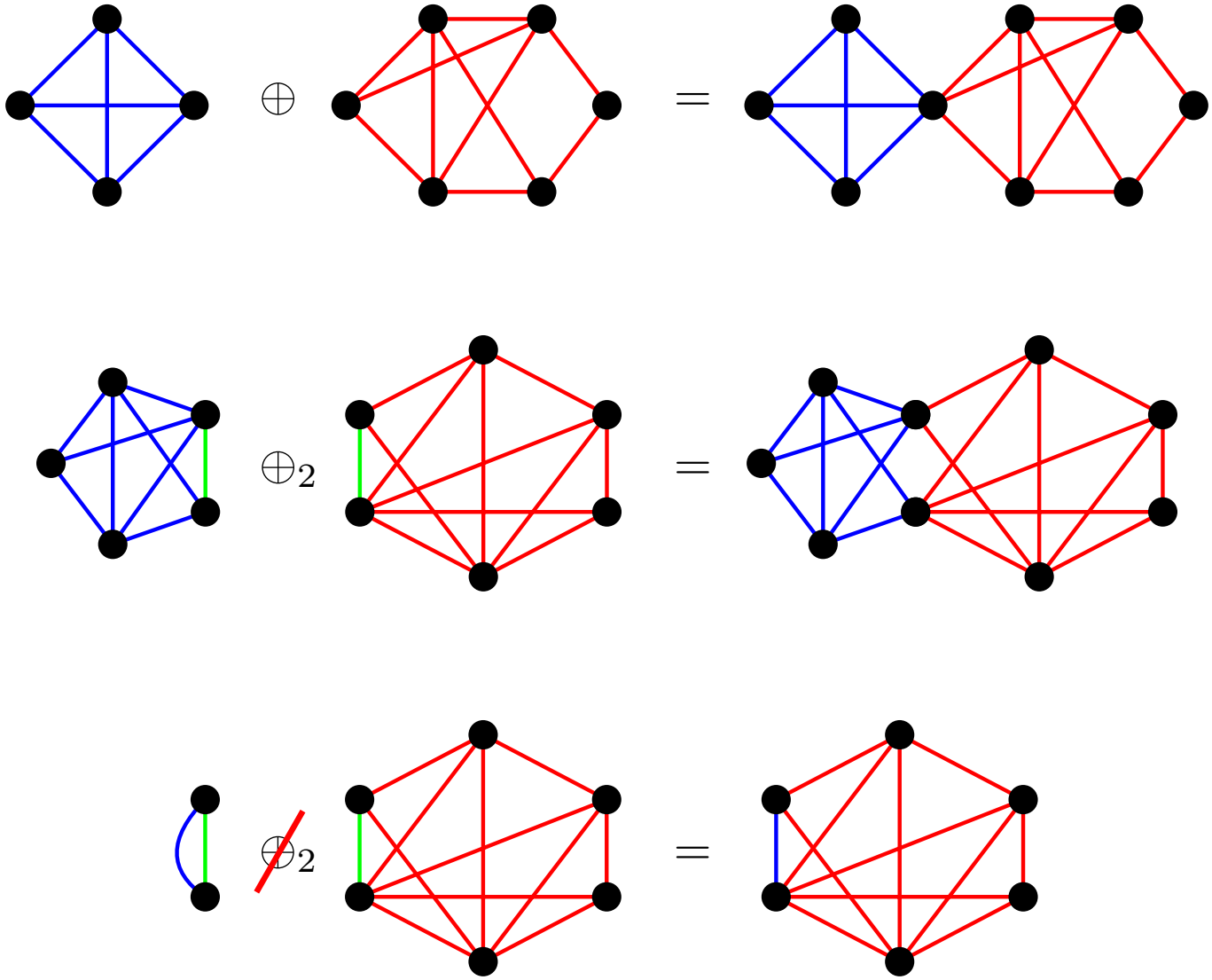
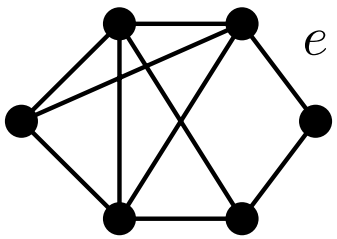


Connected matroids

Theorem (Tutte, 1966)

M 2-connected $\implies$ $M \backslash e$ or M/e is 2-connected for all $e \in E(M)$



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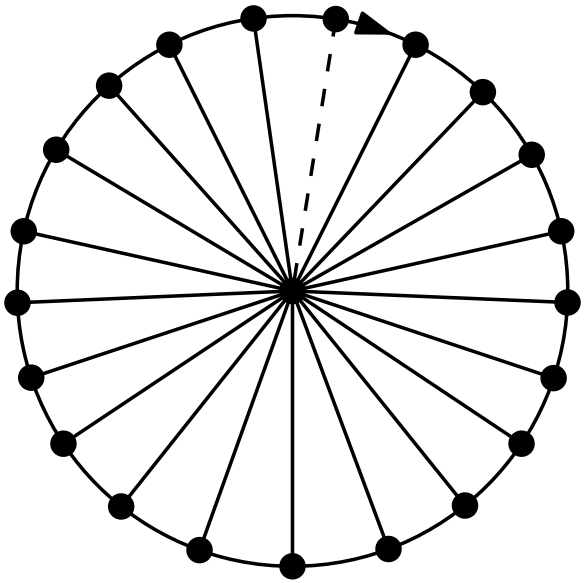
M is a wheel or whirl



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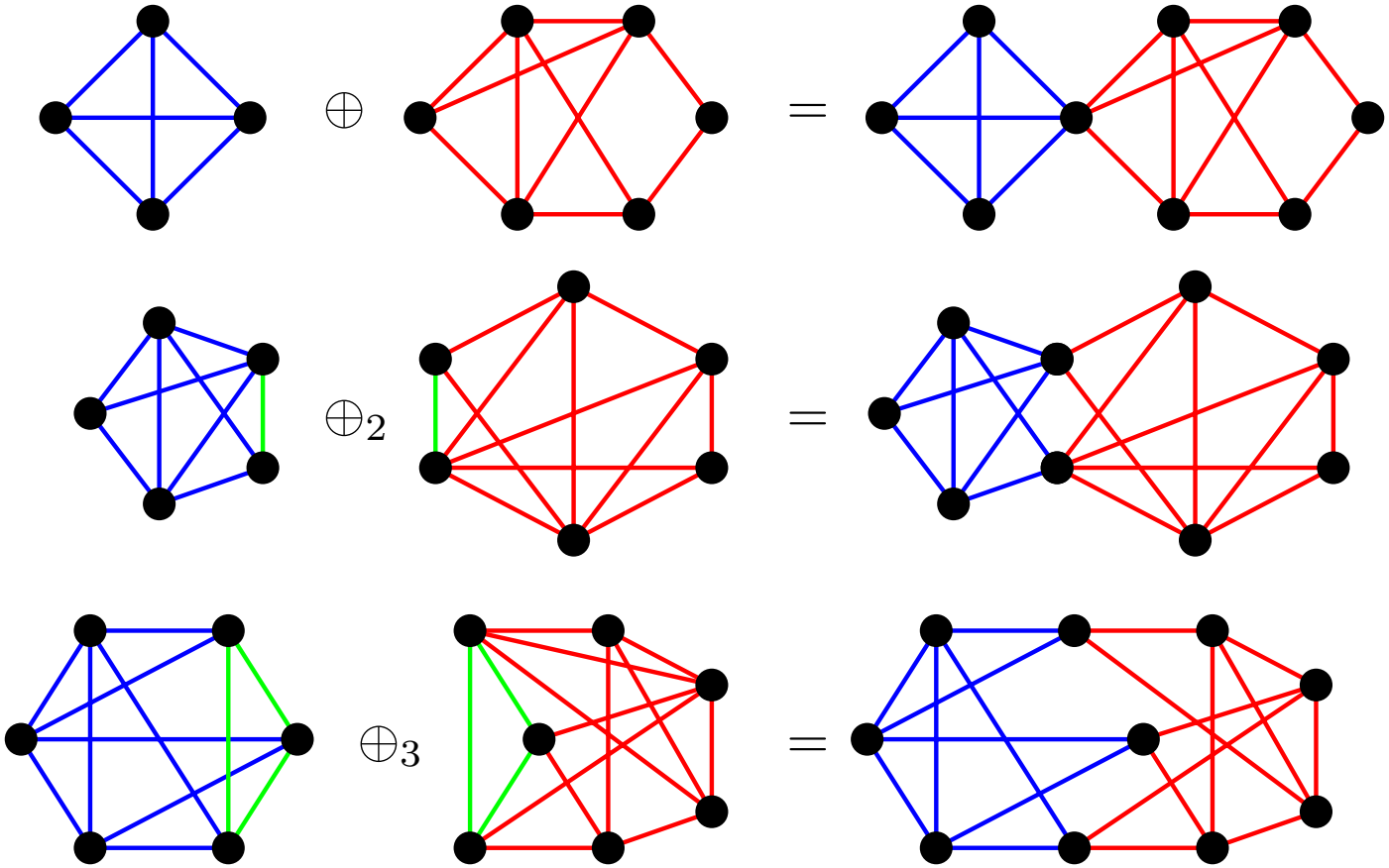

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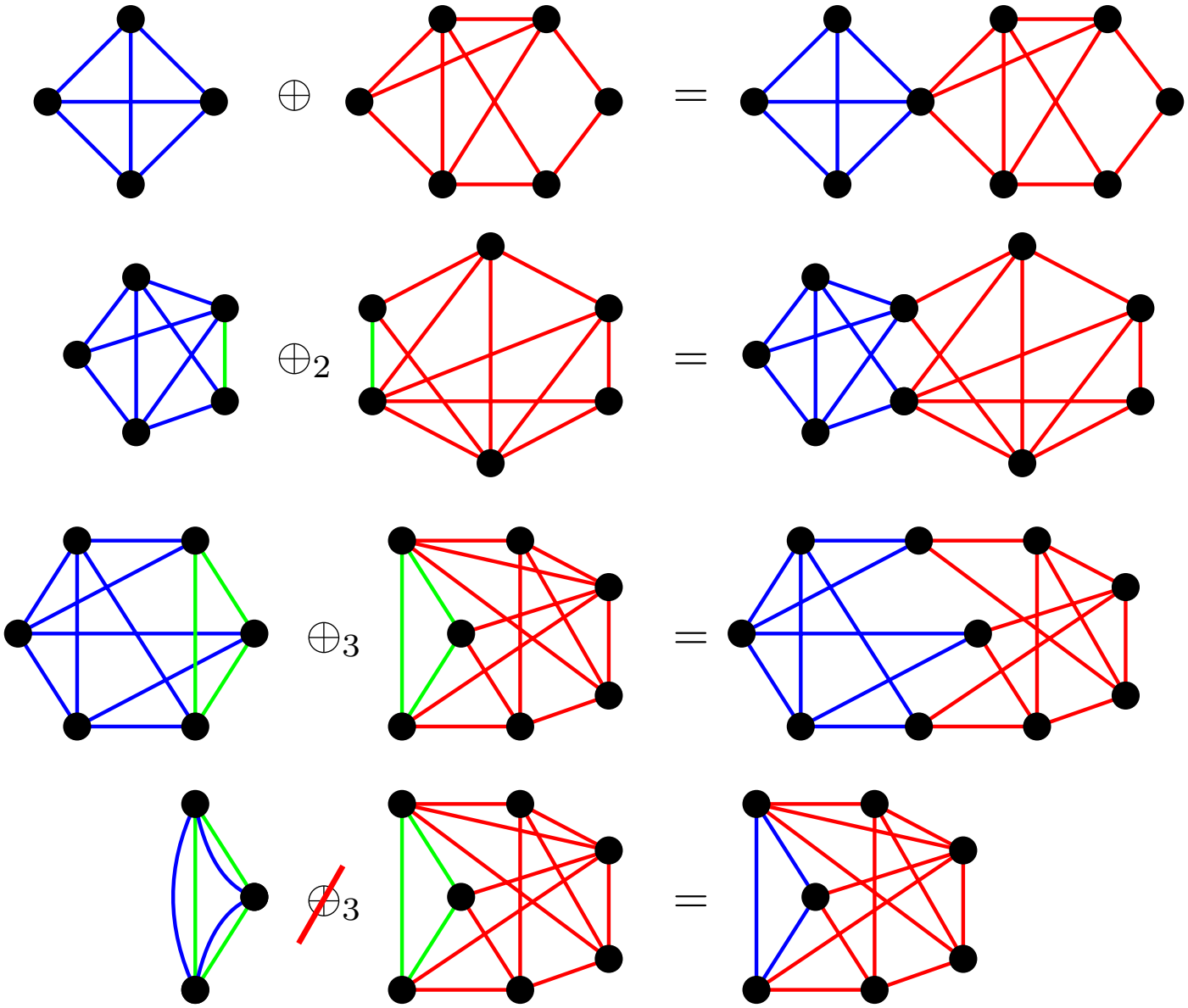

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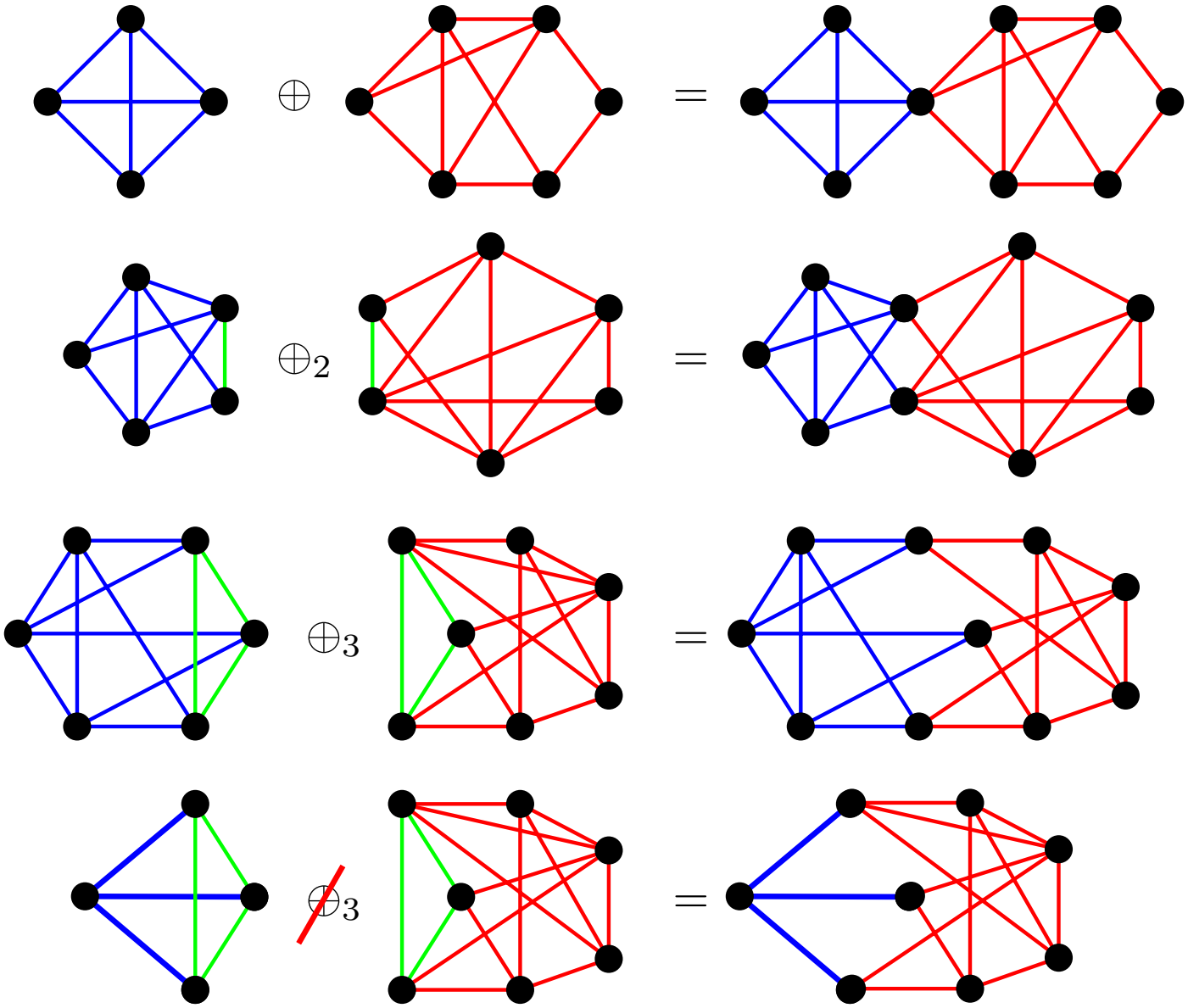

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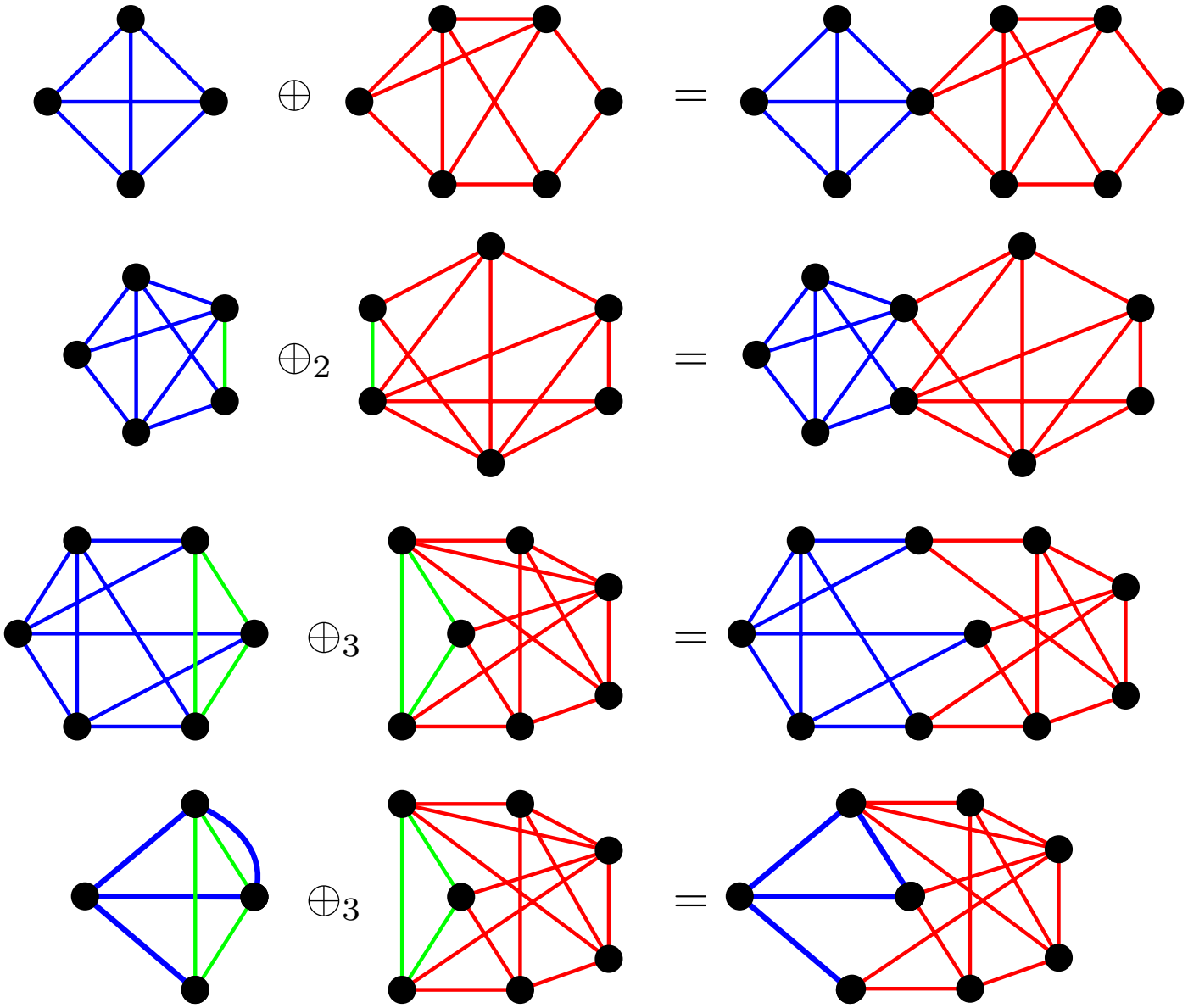
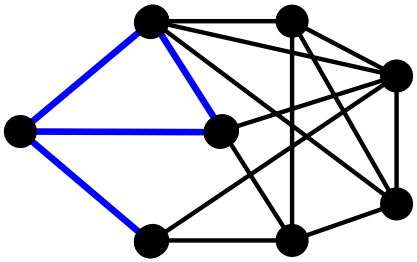


Connectivity

internally 4-connected violator, AKA 4-fan



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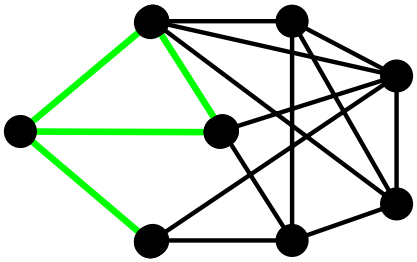

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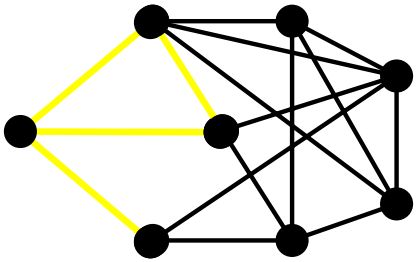

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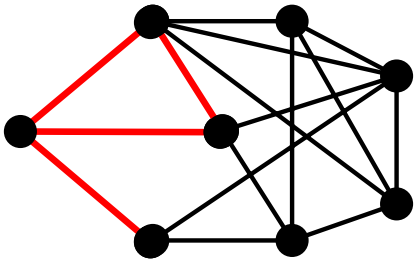

Internally 4-connected binary matroids

Theorem (2011)

Let M be an internally 4-connected binary matroid. Then M has an internally 4-connected minor M' with

$$1 \leq |E(M) - E(M')| \leq 3$$

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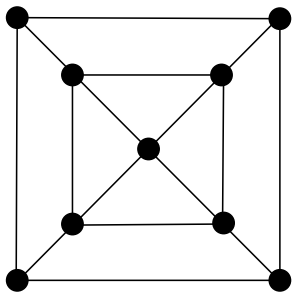
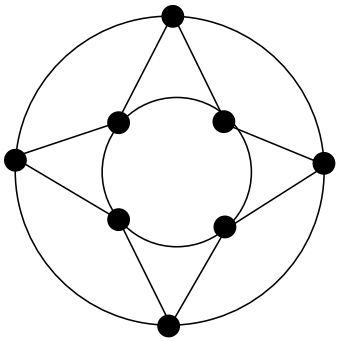
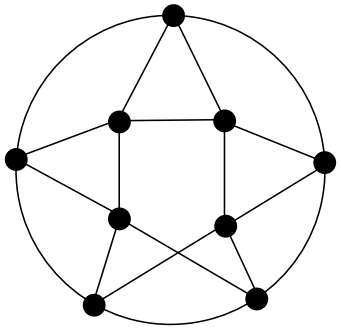
unless M or M^* is the cycle matroid of

(i) a terrahawk; or (ii) a planar or Möbius quartic ladder.



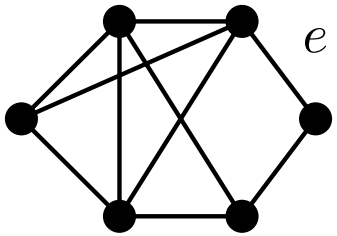




Splitter theorem

Theorem (Brylawski 1972, Seymour 1977)

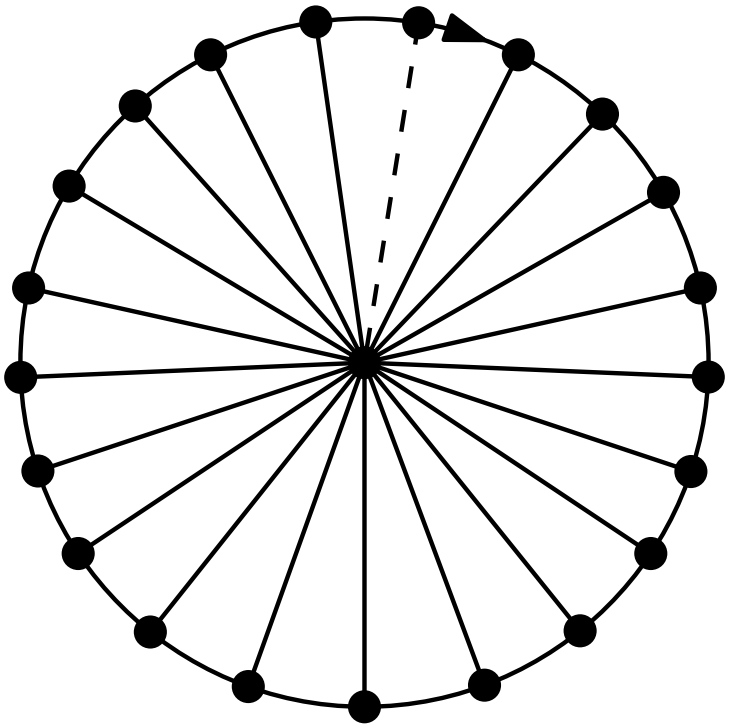
M, N 2-connected, $N \preceq M$, and $e \in E(M) - E(N) \implies M \backslash e$ or M/e is 2-connected with N as a minor



3-connected matroids

Theorem (Seymour's Splitter Theorem)

M & N 3-connected and $N \precneqq M$, where $|E(N)| \geq 4$ and, if N is a wheel/whirl, then M has no larger wheel/whirl-minor $\implies$ $\exists M'$ where $M' \precneqq M$ and $N \preceq M'$ and $|E(M)| - |E(M')| = 1$



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Graphs are binary matroids, what do we know for graphs?
Johnson and Thomas (2001)

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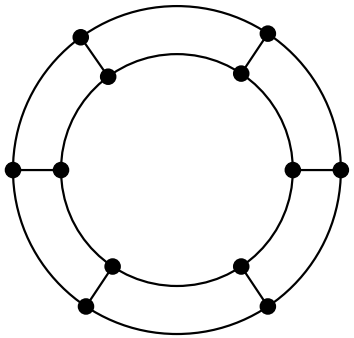
What did Johnson and Thomas show, exactly?

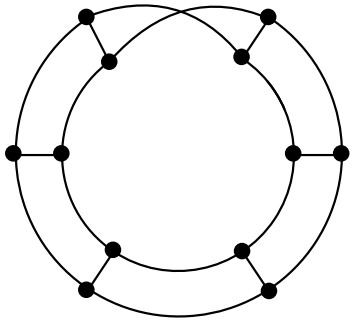
Thor Johnson and Robin Thomas

Splitter Theorem for internally 4-connected graphs
(Generating internally 4-connected graphs, JCTB 2001)

Thor Johnson and Robin Thomas



CPL_k



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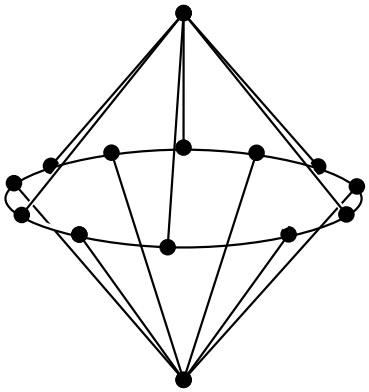
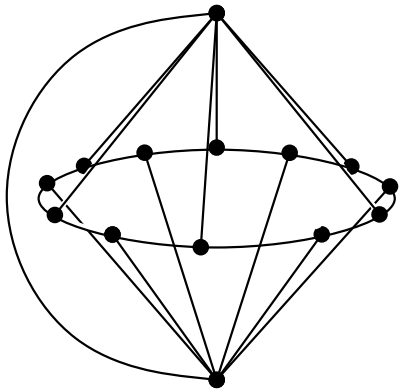


$\mathrm{CPB\text{-}W}_k$



$\mathrm{CMB\text{-}W}_k$

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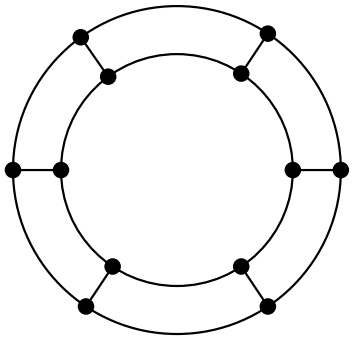

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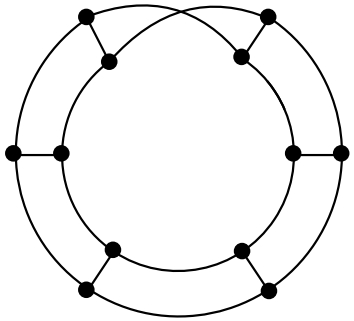

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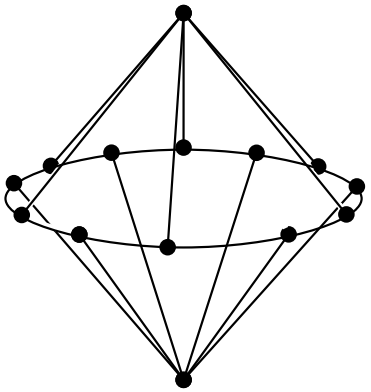

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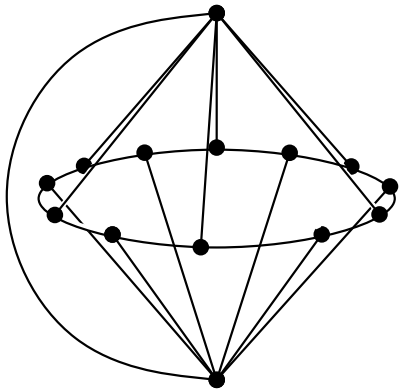

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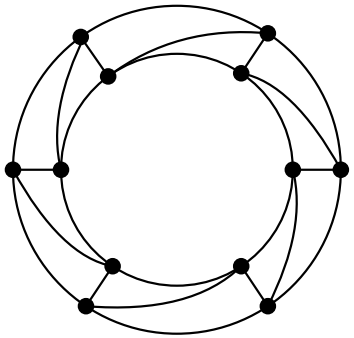
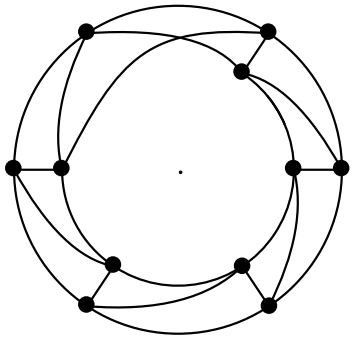
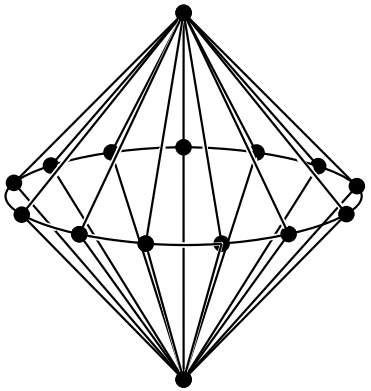


QPL_k



QML_{k+1}



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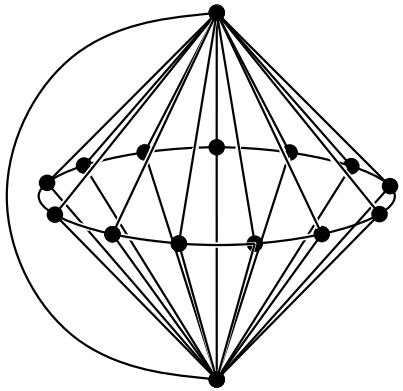

$\mathrm{QMB\text{-}W}_k$

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Theorem (Theorem 1)

G & H internally 4-connected graphs and $H \precsim\!\!\!\!\neq G$ and, if H is a CPL_k

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G & H internally 4-connected graphs and $H \precneqq G$ and, if H is a $CPL_k/CML_k/CPB\text{-}W_k$

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G & H internally 4-connected graphs and $H \precsim\!\!\!\!\!\neq G$ and, if H is a $CPL_k/CML_k/CPB\text{-}W_k/CMB\text{-}W_k$, then G has no QPL_k

Thor Johnson and Robin Thomas

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Thor Johnson and Robin Thomas

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(i) *$H'/e = H$ for some edge e; or*

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(ii) *$E(H') = E(H) \cup \{e_1, e_2, \ldots, e_t\}$, where $(V(H'), E(H) \cup \{e_1, e_2, \ldots, e_i\})$ has at most one 4-fan for all $i \in \{1, 2, \ldots, t-1\}$ and $(V(H'), E(H) \cup \{e_1, e_2, \ldots, e_t\})$ is internally 4-connected; or*

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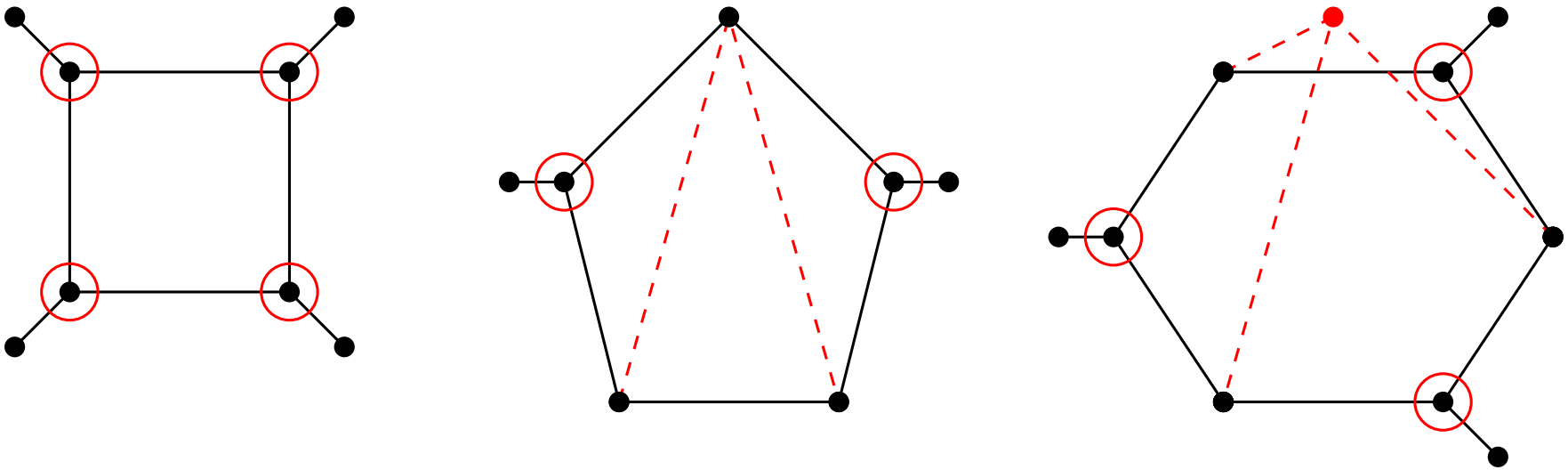

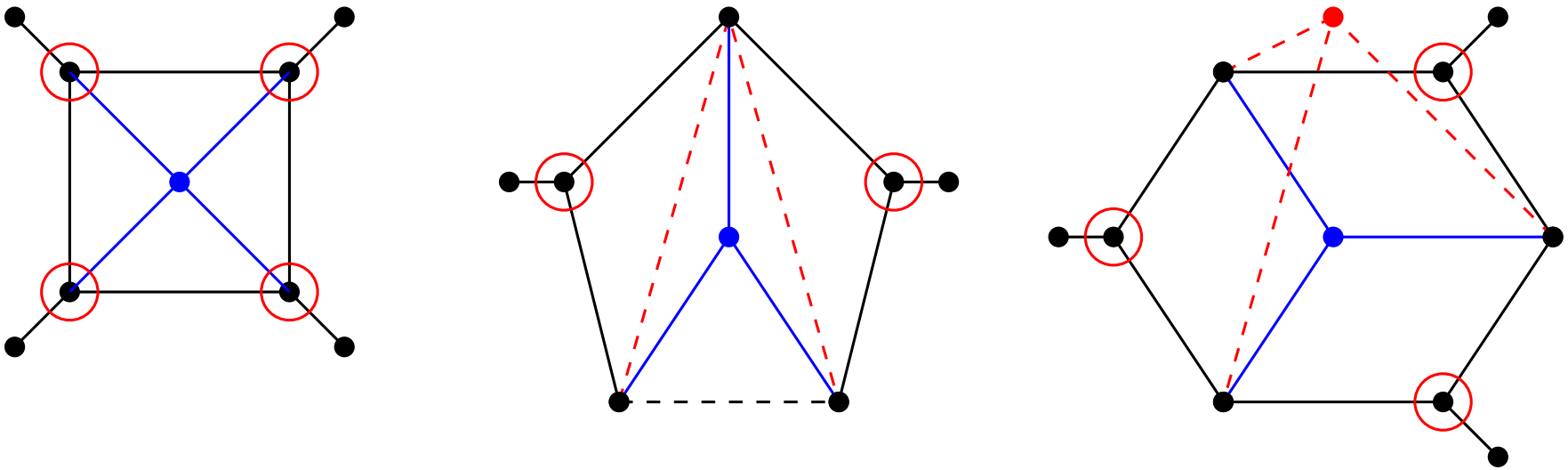

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Theorem (Theorem 2)

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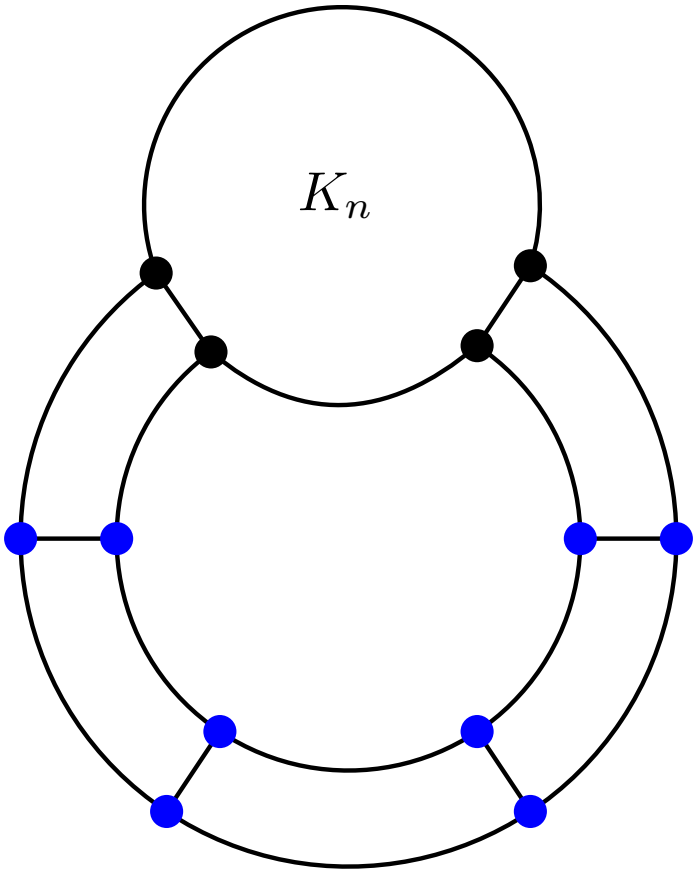
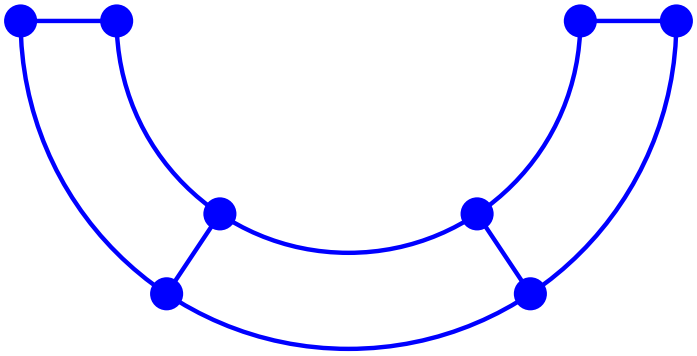
(ii) *$H' \backslash e = H$ for some edge e; or*

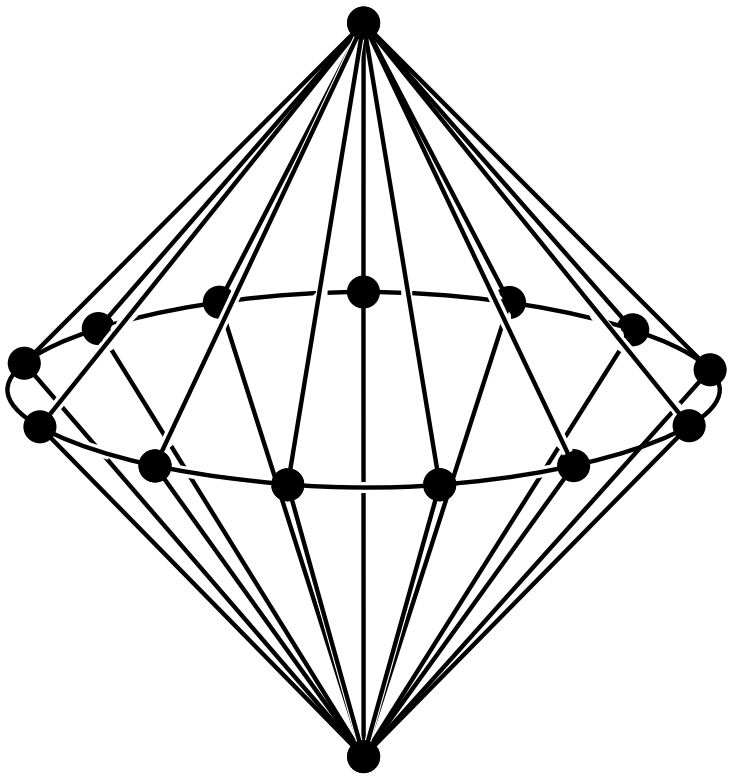
(iii) *H' is obtained from H by a quadrangular, pentagonal, or hexagonal extension.*


K_n



Internally 4-connected binary matroids

How can we get an internally 4-connected minor of this?



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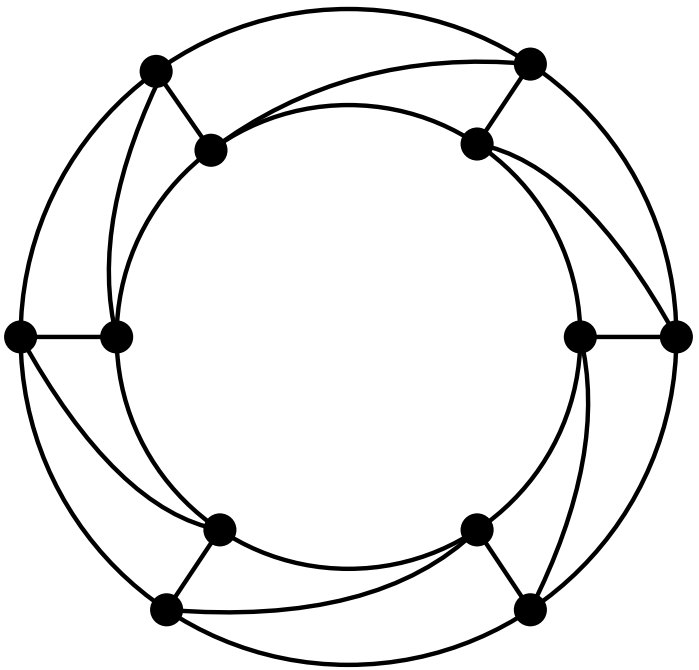

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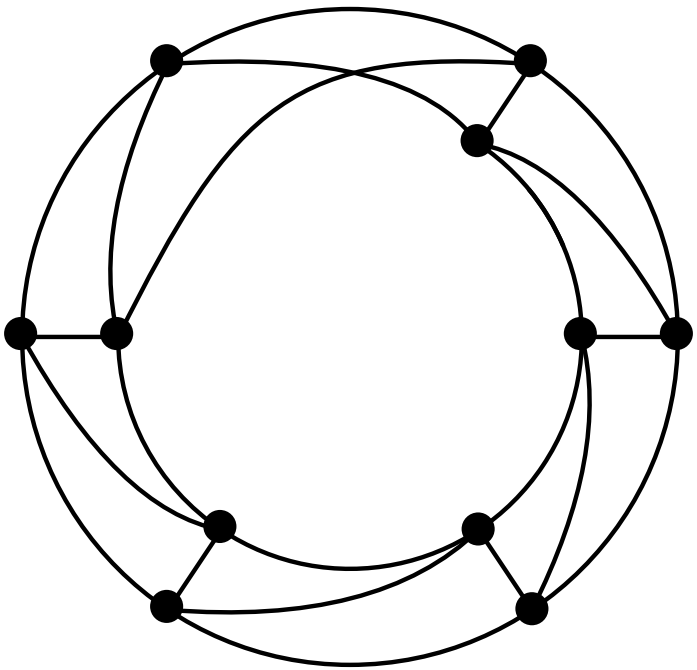

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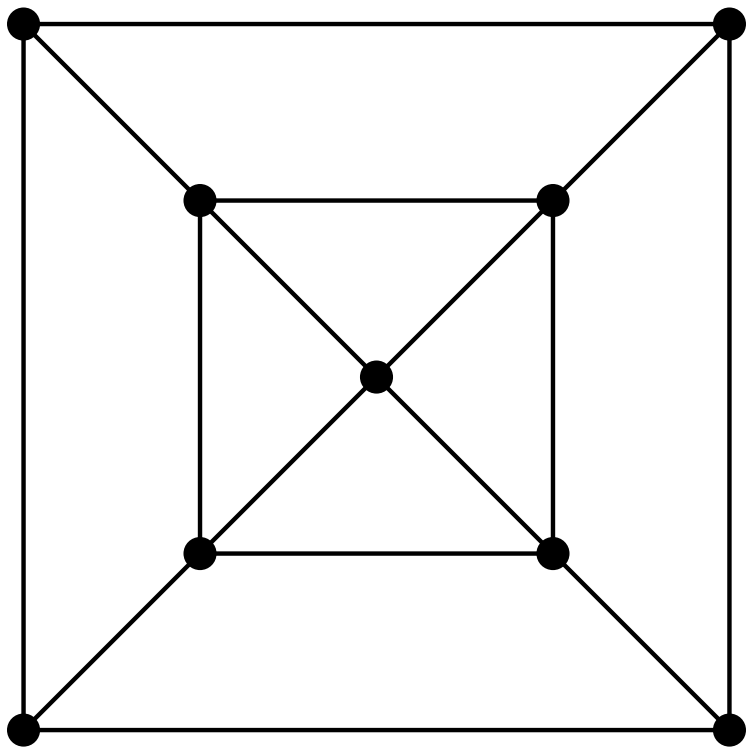

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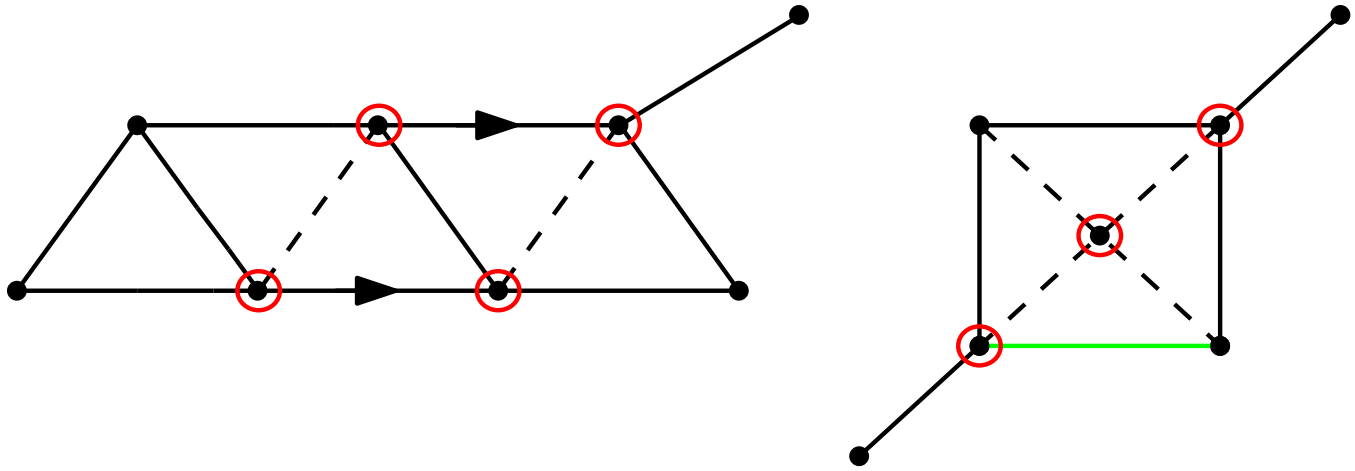

get M' from M by a central cocircuit deletion in a good augmented 4-wheel or a ladder-compression move; or ...

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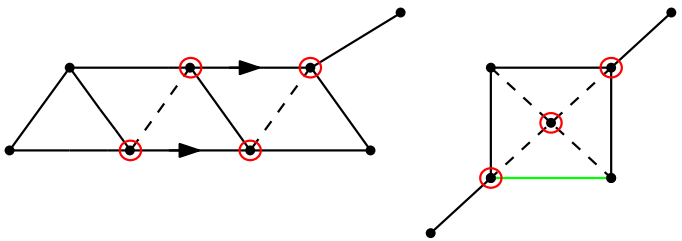
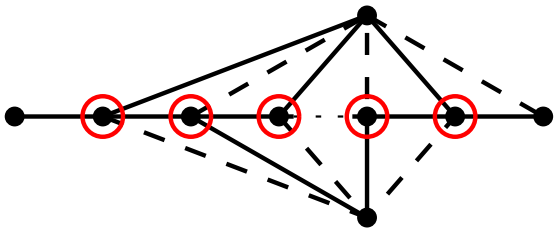


(ii) *get M' from M by trimming a double fan, a bowtie ring, or a "ladder;" or ...*



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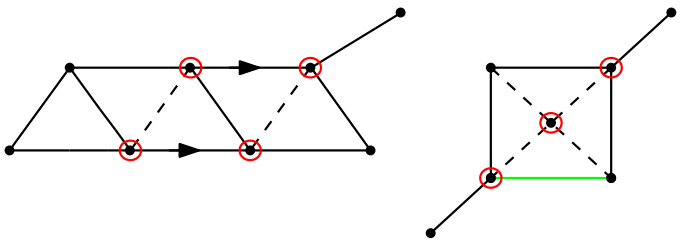

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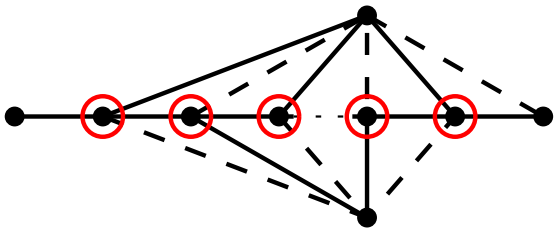

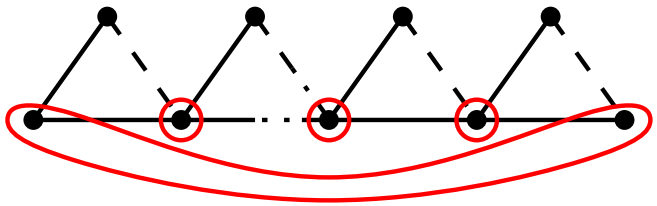

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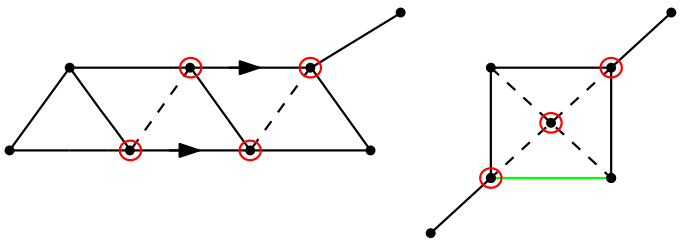

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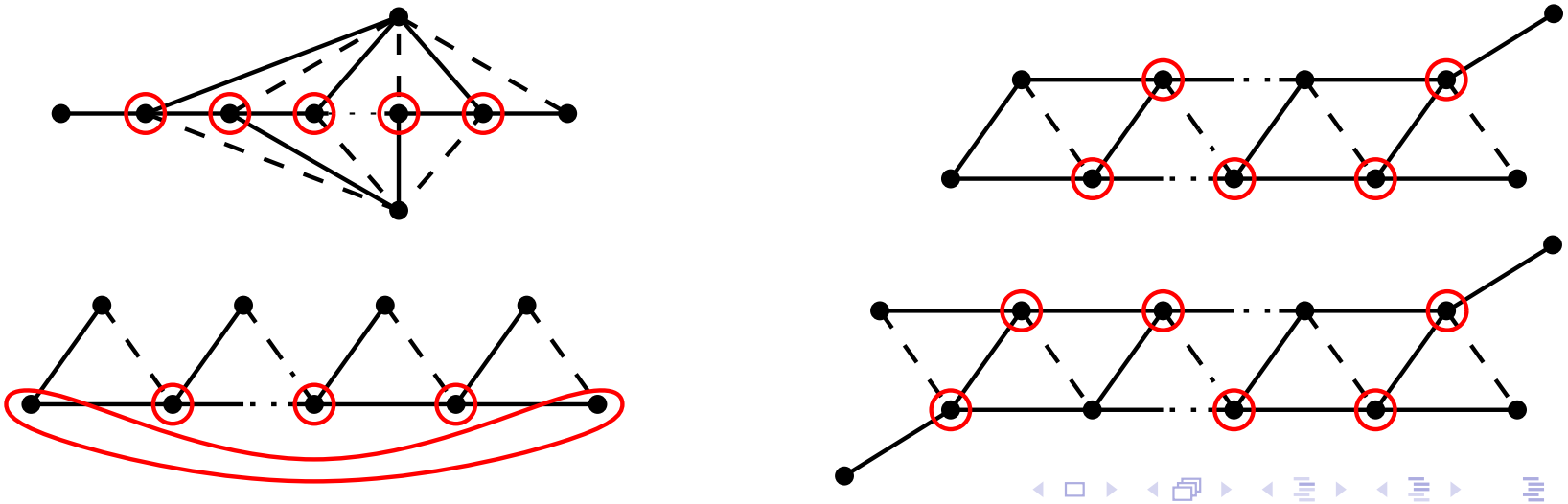

Internally 4-connected binary matroids

(iii) get M' from M by an enhanced-ladder move or lobster move; or ...

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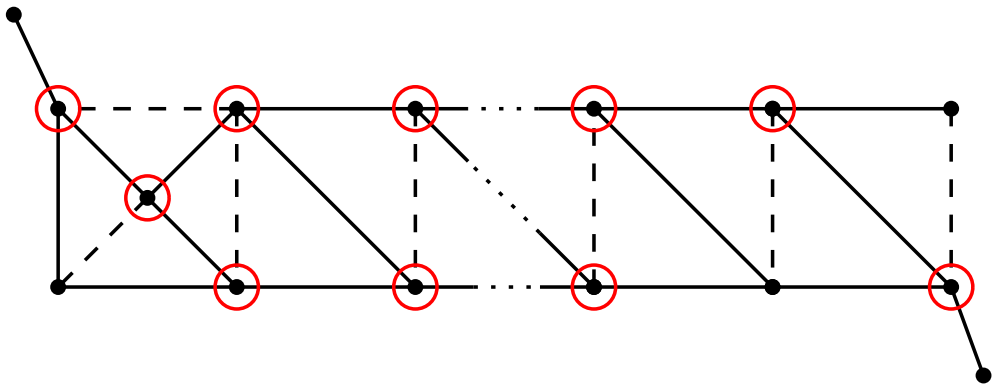

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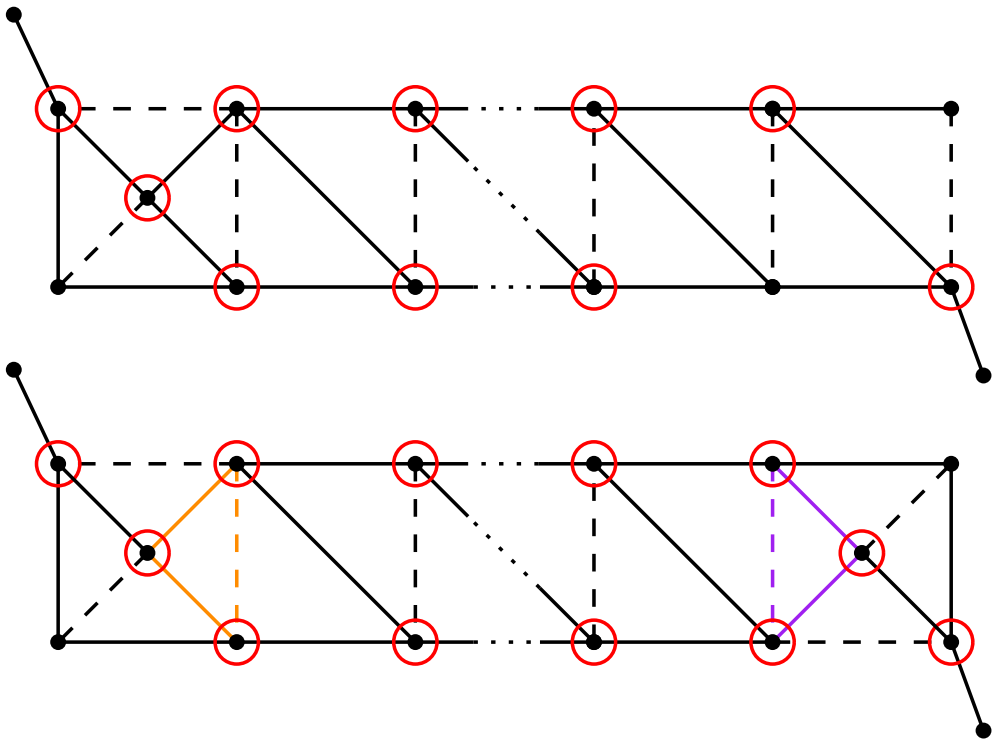

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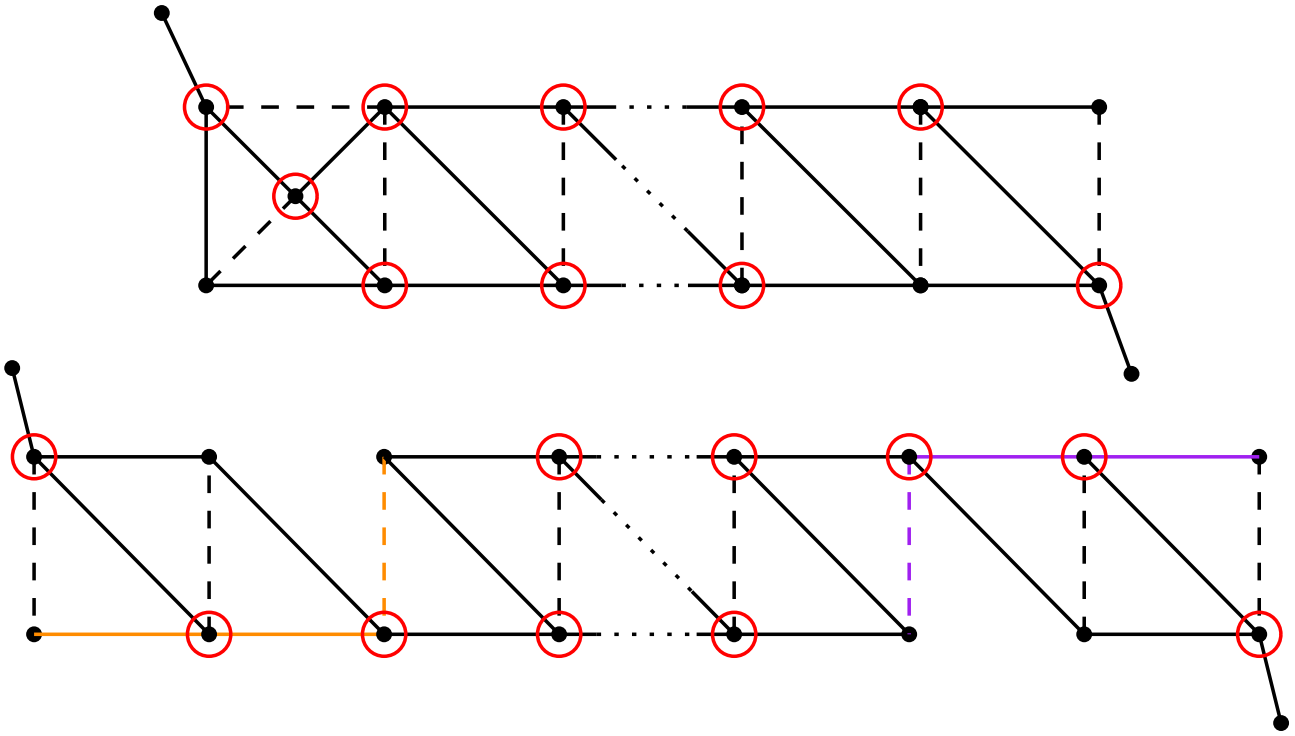

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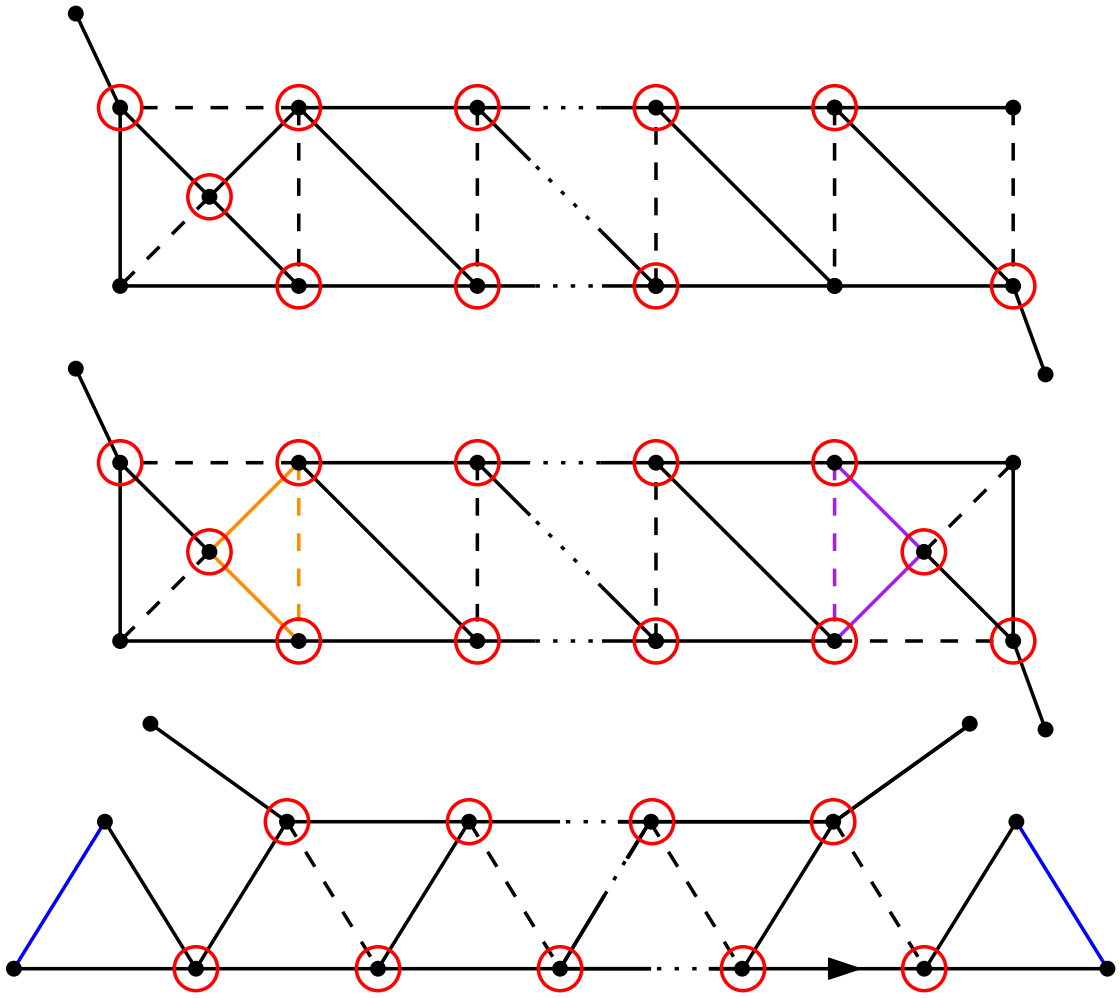

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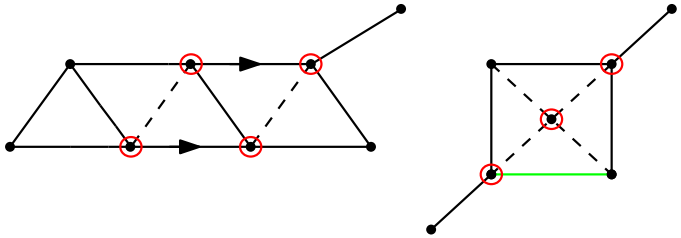
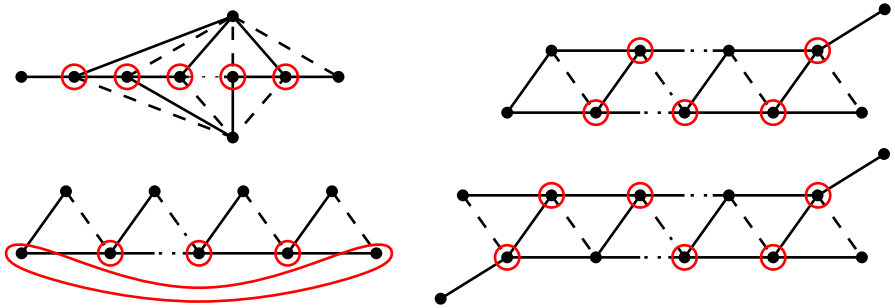
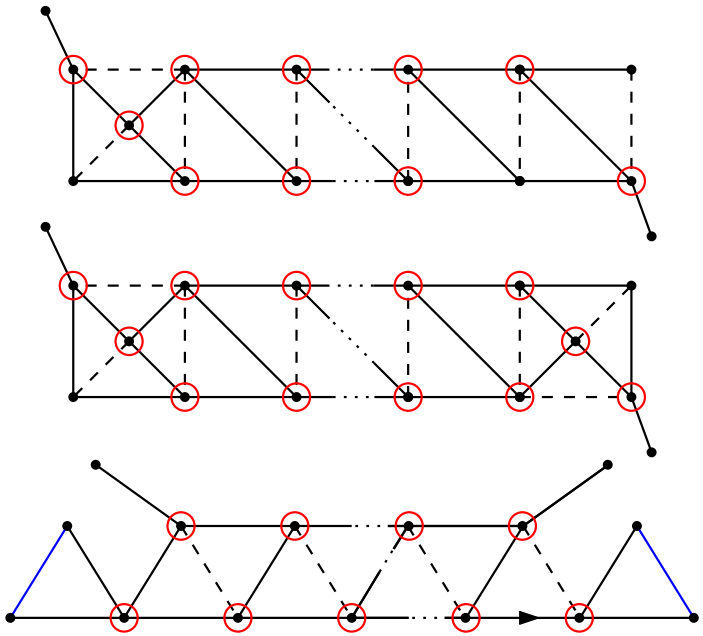


(ii)



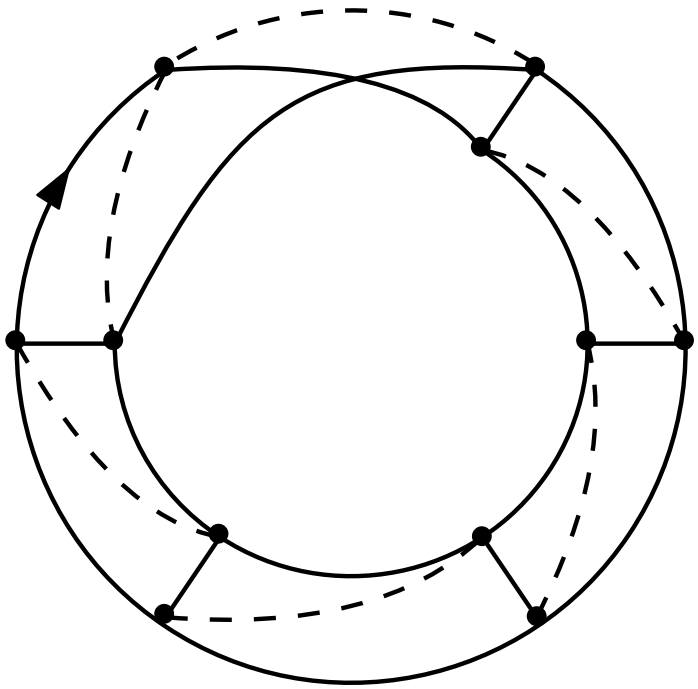
(iii)

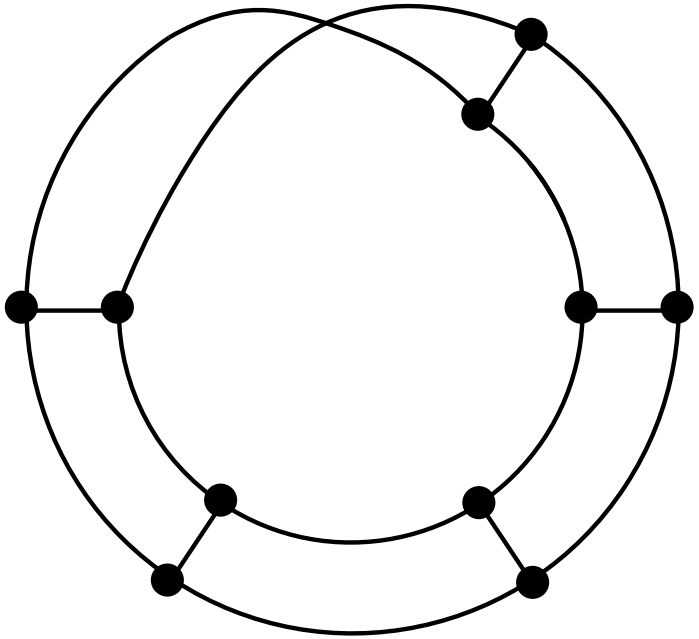


or ...

Internally 4-connected binary matroids

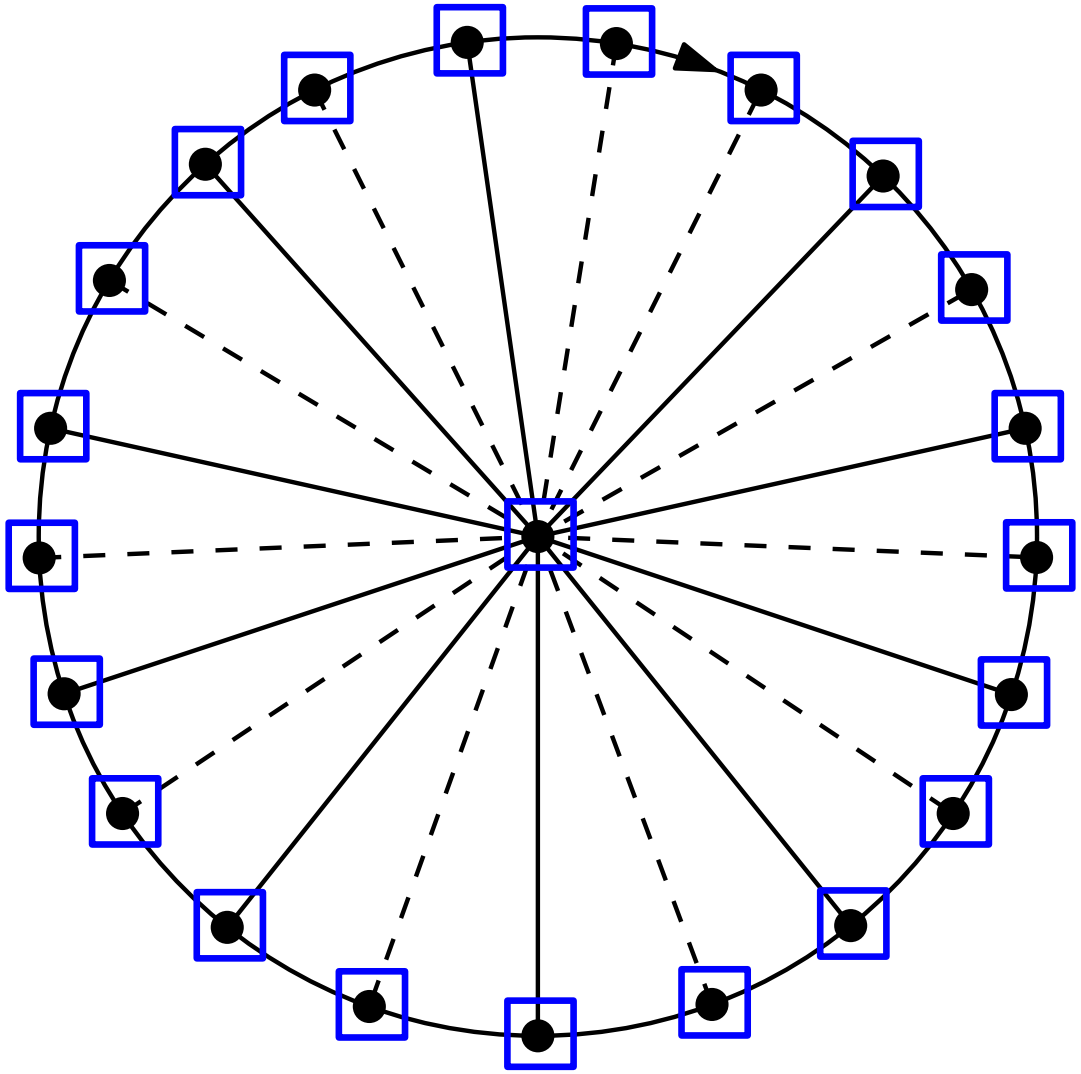
(iv) (M, N) is (QML,CML) or ...





Internally 4-connected binary matroids

(iv) (M, N) is (QML,CML) or (T*MM,TMM); or ...



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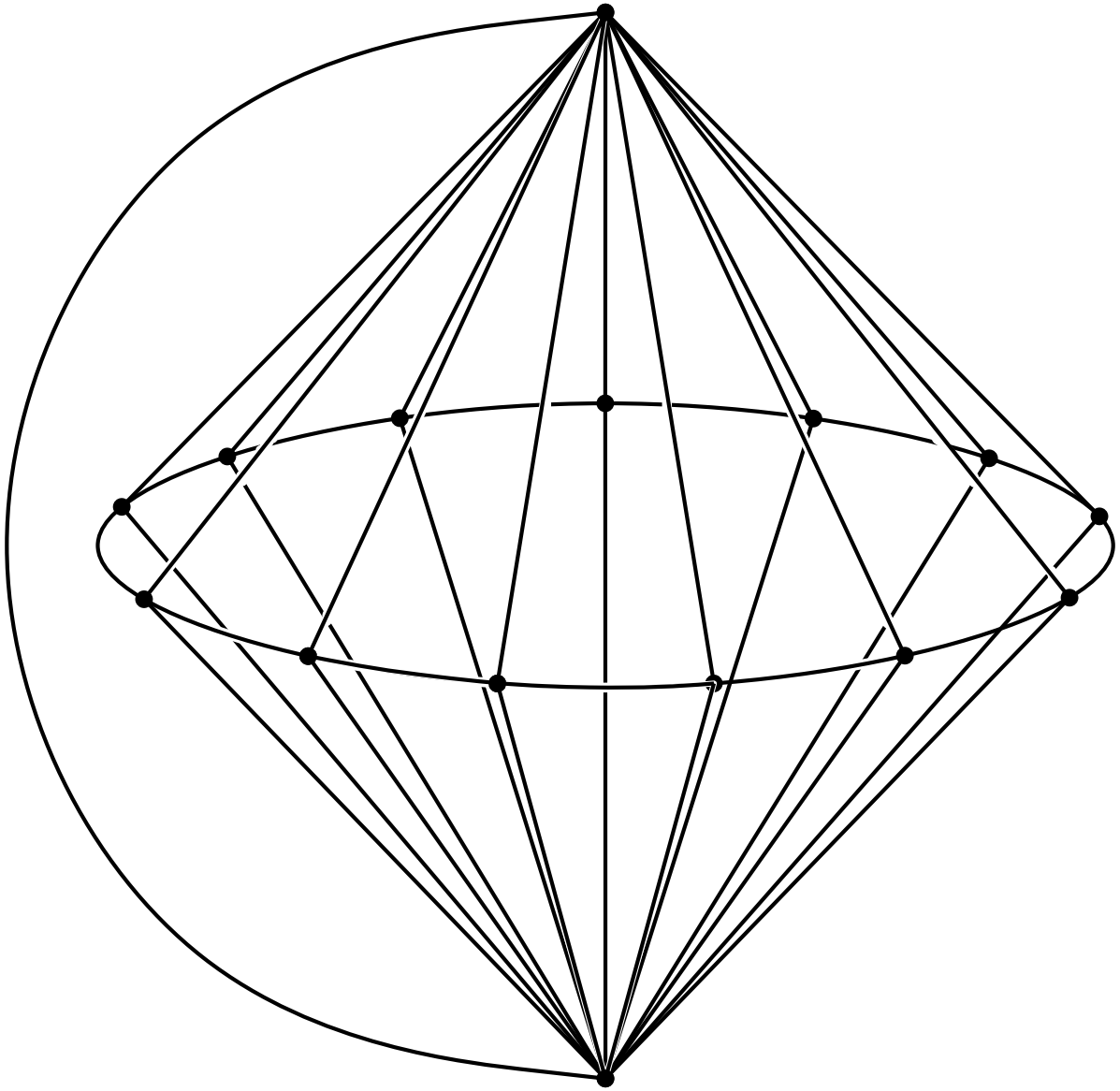

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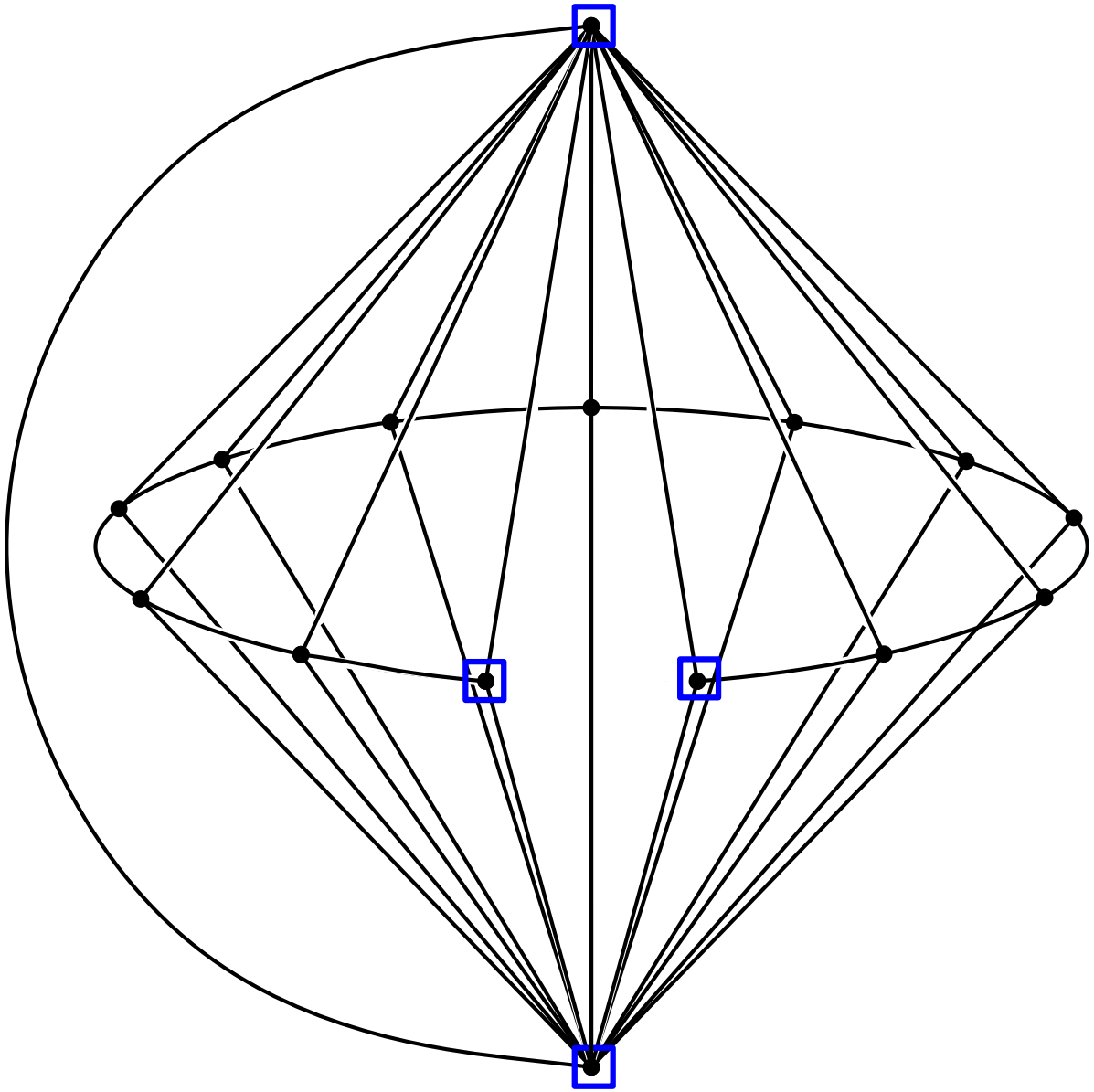

Internally 4-connected binary matroids

(v) (M, N) is one of 31 *interesting pairs* (A splitter theorem for internally 4-connected binary matroids: small matroids, arXiv).

interesting pair (n.) $|E(M)| \leq 15$ and M and N are internally 4-connected, but no proper minor of M with a proper N-minor is internally 4-connected

Splitter Theorem for internally 4-connected binary matroids

Theorem

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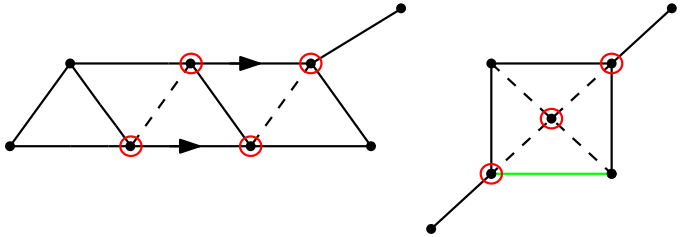
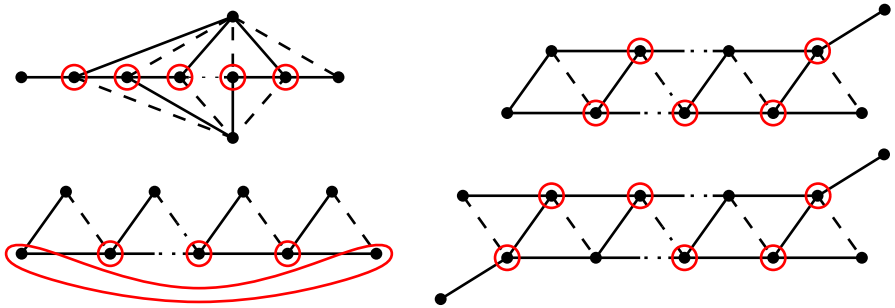
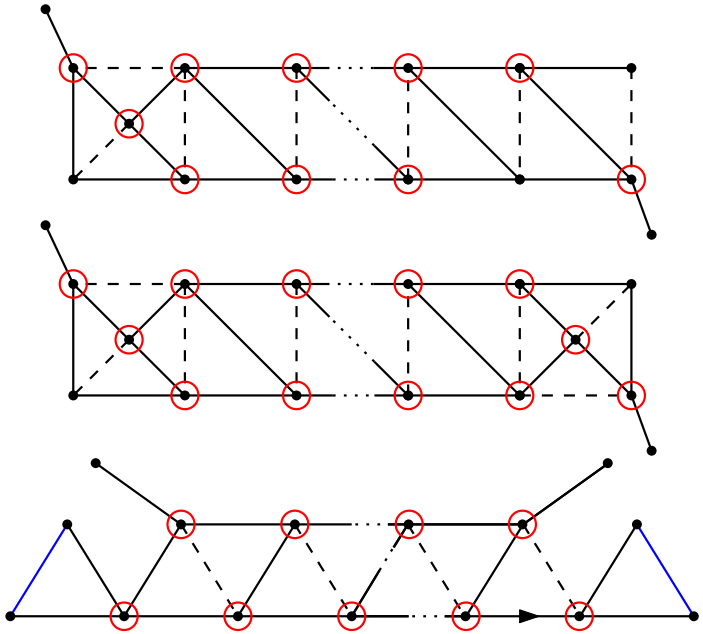
(i)



(ii)



(iii)



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Chun, Mayhew, & Oxley splitter theorem

(i) give no restriction for N; and

(ii) have several highly-structured unbounded moves from M to internally 4-connected intermediate matroid; and

(iii) have nice bounded (≤ 4) moves from M to internally 4-connected intermediate matroi; and

(iv) is useful for finding excluded minor results.

Splitter Theorem for internally 4-connected binary matroids

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M, N internally 4-connected binary matroids & $N \lneq M \implies$ $\exists M'$, internally 4-connected, where $N \preceq M' \lneq M$ & $|E(M)| - |E(M')| \leq 3$ or M and N are Möbius ladders or Möbius matroids or comprise an interesting pair or, (up to duality), M' is obtained by one of these moves in M:

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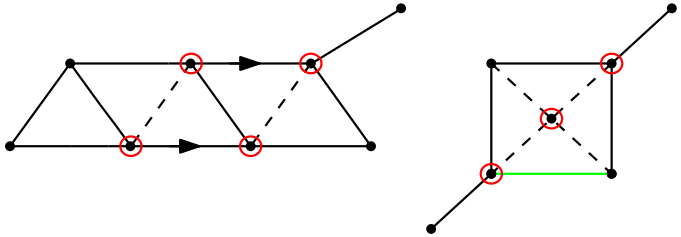

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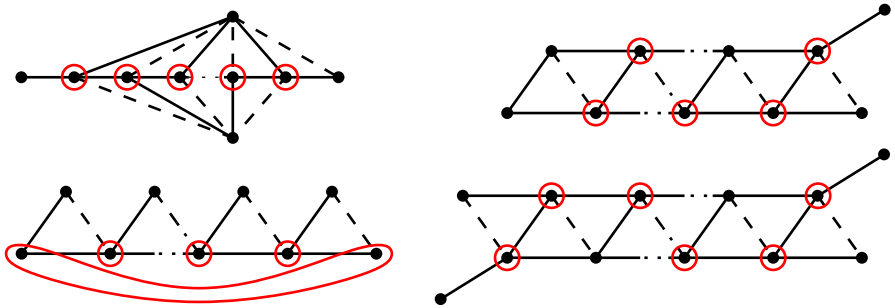

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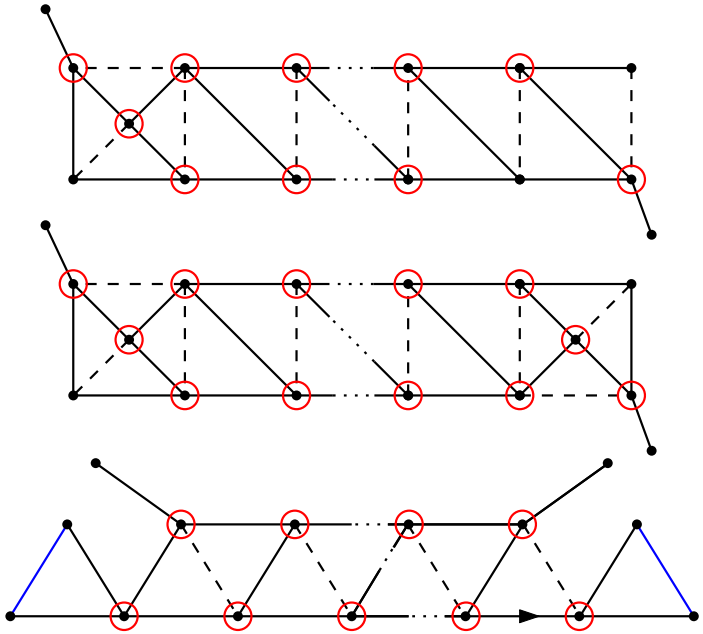
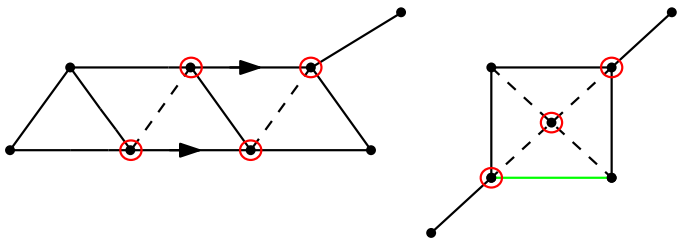


Cor: Splitter Theorem for internally 4-connected graphs

Theorem

G, H internally 4-connected graphs & $H \lneq G \implies$ $\exists G'$, internally 4-connected, where $H \preceq G' \lneq G$ & $|E(G)| - |E(G')| \leq 3$ or G and H are Möbius ladders or comprise one of 12 interesting pairs or, (up to duality), G' is obtained by one of these moves in G:

(i)



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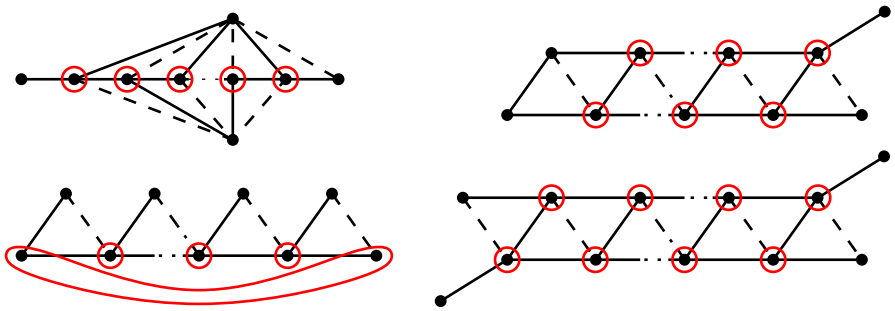

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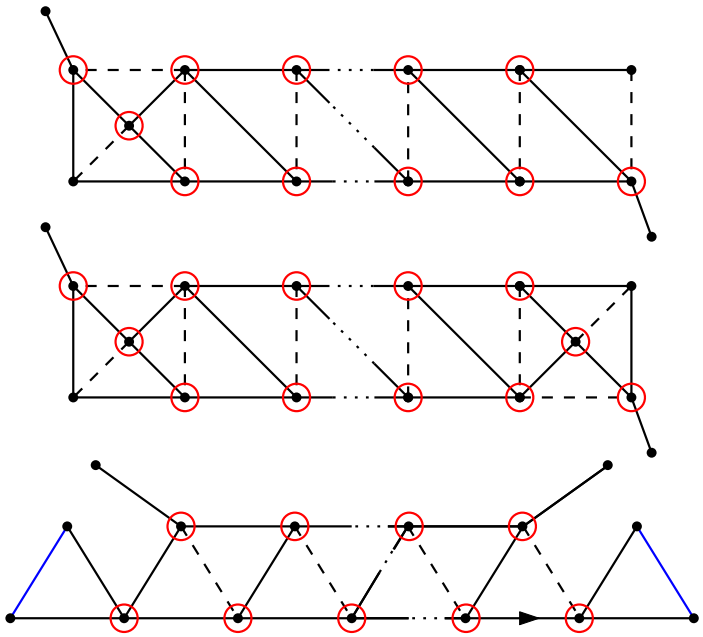
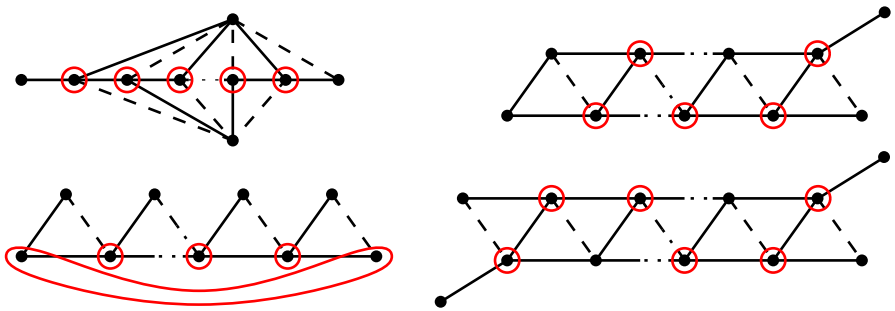


Cor: back to internally 4-connected binary matroids

Theorem

M, N internally 4-connected binary matroids & $N \lneq M \implies$ $\exists M'$, internally 4-connected, where $N \preceq M' \lneq M$ & $|E(M)| - |E(M')| \leq 6$ or M and N are Möbius ladders or Möbius matroids or, (up to duality), M' is obtained by one of these moves in M:

(i)



(ii)