LINKAGES IN A DIRECTED GRAPH WITH PARITY RESTRICTIONS

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Abstract. Given a digraph D and a graph G with common vertex set V and a set T of terminals in V, we give a necessary and sufficient condition for the existence of a k-edge matching of G whose vertex set is linked to T by vertex-disjoint dipaths of D. The result we obtain is a common generalisation of the Tutte-Berge formula and Menger's Theorem.

1. Introduction

If $D = (V, A)$ is a digraph and S and T are subsets of V, we say that S is *T-linked in D* if there is a collection of $|S|$ vertex-disjoint directed paths from S to T in D. We write $D^{S,T}$ for the digraph obtained from D by removing all arcs with head in S or tail in T; note that S is T-linked in D if and only if it is T-linked in $D^{S,T}$. Given a partition $\mathcal{P}$ of the vertex set of a graph or digraph, an edge or arc *crosses* $\mathcal{P}$ if its ends lie in different blocks of $\mathcal{P}$. We prove the following:

Theorem 1.1. *Let $G = (V, E)$ be a graph and $D = (V, A)$ be a digraph. If $T \subseteq V$ and $k \geq 0$, then exactly one of the following holds:*

(1) G has a k-edge matching M so that $V(M)$ is T-linked in D.
(2) There are sets S', T' with $S' \subseteq V$ and $T \subseteq T' \subseteq V$ and a partition $\mathcal{P}$ of V such that no edge of $G - S'$ or arc of $D^{S',T'}$ crosses $\mathcal{P}$ and $\sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2} \left(|P \cap S'| + |P \cap T'|\right) \right\rfloor < k$.

The T-linked subsets of V are the independent sets of a representable matroid known as a *strict gammoid* [2 p.659], so the above can be stated as a matroid matching problem. Tong, Lawler and Vazirani [3] observed that this problem can be reduced to a graph matching problem in an auxiliary graph H. We derive Theorem 1.1 by applying the Tutte-Berge formula (which is straightforward to recover by setting $T = V$ in the above) to H. In fact, we prove a slightly more general result:

This research was partially supported by a grant from the Office of Naval Research [N00014-12-1-0031].

Theorem 1.2. *Let $G = (V, E)$ be a graph and $D = (V, A)$ be a digraph. If $S, T \subseteq V$ and $k \geq 0$ then exactly one of the following holds:*

(1) There is a matching M of G and a set $X \subseteq S \setminus V(M)$ such that $|M| + |X| = k$ and $V(M) \cup X$ is T-linked in D.
(2) There are sets S', T' with $S \subseteq S' \subseteq V$ and $T \subseteq T' \subseteq V$ and a partition $\mathcal{P}$ of V such that no edge of $G - S'$ or arc of $D^{S',T'}$ crosses $\mathcal{P}$ and $\sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2} (|P \cap S'| + |P \cap T'|) \right\rfloor < k$.

It is fairly easy to check (and will be proved later) that the summation in (2) for any admissible $\mathcal{P}$ is an upper bound for the size of $|M| + |X|$ as in (1). Setting $S = \varnothing$ yields Theorem 1.1; we consider other applications, including a derivation of Menger's theorem for vertex-disjoint paths in a digraph, in Section 4.

2. Preliminaries

All graphs and digraphs are simple. For $X \subseteq V(G)$ we write $G - X$ for the graph obtained by deleting the vertices in X. For a matching M, we write $V(M)$ for the set of vertices saturated by M. For a digraph $D = (V, A)$ and subsets S and T of V, an *(S,T)-linkage* in D is a set $\mathcal{Q}$ of $|S|$ vertex-disjoint (S, T)-dipaths in D. We phrase the following well-known result [2 p. 413] in a convenient form.

Theorem 2.1 (Tutte-Berge Formula)**.** *If $G = (V, E)$ is a graph, then*

$$\nu(G) = \min_{Z \subseteq V} \left(|Z| + \sum_{C} \left\lfloor \tfrac{1}{2} |V(C)| \right\rfloor \right),$$

where the summation is taken over the components C of $G - Z$.

As usual, $\nu(G)$ denotes the size of a maximum matching of G; we now extend this notation to deal with linked matchings. If $G = (V, E)$ is a graph with a set $S \subseteq V$ of 'roots' and $D = (V, A)$ is a digraph with a set $T \subseteq V$ of 'terminals', then we write $\nu(G, D; S, T)$ for the maximum of $|M| + |X|$ such that M is a matching of G, $X \subseteq S \setminus V(M)$ and $X \cup V(M)$ is T-linked in D.

We now define the auxiliary graph to which Theorem 2.1 will be applied. Let $G = (V, E)$ be a graph, $D = (V, A)$ be a digraph, and S, T be subsets of V. Let $\widehat{V} = \{\hat{v} : v \in V\}$ be a disjoint copy of V and let $\widehat{U} = \{\hat{u} : u \in U\}$ for each $U \subseteq V$. Let $F = \{\hat{u}v : (u, v) \in A\} \cup \{u\hat{u} : u \in V\} \cup E$. Let $H(G, D; S, T) = (V \cup \widehat{V}, F) - (S \cup \widehat{T})$; note that $V(H) = V \cup \widehat{V} \setminus \Big((T \setminus S) \cup (\widehat{S} \setminus \widehat{T}) \Big)$ and $|V(H)| = 2|V| - |S| - |T|$.

If $P = (v_0, v_1, \ldots, v_j)$ is a dipath of D with end vertex in T and no internal vertex in $S \cup T$, then $\mu_H(P)$ will denote the set of edges

$\{\widehat{v_0}v_1, \widehat{v_1}v_2, \ldots, \widehat{v_{j-1}}v_j\}$. Note that this is a matching of H saturating exactly $\{\widehat{v_0}, v_1, \widehat{v_1}, \ldots, v_{j-1}, \widehat{v_{j-1}}, v_j\}$. If $v_0 = v_j$ then $\mu_H(P)$ is empty.

3. The Proof

We first show that computing $\nu(G, D; S, T)$ can be reduced to computing $\nu(H)$ for the auxiliary graph H.

Lemma 3.1. *Let $G = (V, E)$ be a graph, $D = (V, A)$ be a digraph, and S and T be subsets of V. If $H = H(G, D; S, T)$ then $\nu(H) = \nu(G, D; S, T) - |S| - |T| + |V|$.*

Proof. We first argue that $\nu(H) \geq \nu(G, D; S, T) - |S| - |T| + |V|$. Let M be a matching of G and $X \subseteq S - V(M)$ be a set so that $|M| + |X| = \nu(G, D; S, T)$ and $X \cup V(M)$ is T-linked in D. Let $\mathcal{Q}$ be an $(X \cup V(M), T)$-linkage in D; by choosing M, X and $\mathcal{Q}$ so that the total length of the paths in $\mathcal{Q}$ is minimized, we may assume that $\mathcal{Q}$ links every vertex in $S \cap T$ to itself by a trivial path (so $S \cap T \subseteq X$), and that no path in $\mathcal{Q}$ has an internal vertex in $S \cup T$. Let $Y = X \setminus (S \cap T)$ and let $T_0 \subseteq T$ be the set of end vertices of dipaths in $\mathcal{Q}$ with start vertex in Y, so $|T_0| = |Y| + 2|M|$. Let $M_0 = \cup_{P \in \mathcal{Q}} \mu_H(P) \cup E(M)$ and $M_1 = M_0 \cup \{v\widehat{v} : v \in V - (S \cup T \cup V(M_0))\}$. The set M_1 is a matching of H with $V(M_1) = V(H) - ((\widehat{U} \setminus \widehat{T}) \setminus \widehat{Y}) \cup ((S \setminus T) \setminus T_0)$. Therefore

$$\nu(H) \geq |M_1| = \tfrac{1}{2}(|V(H)| - |S \setminus T| + |Y| - |T \setminus S| + |T_0|).$$

Using $|V(H)| = 2|V| - |S| - |T|$, $|T_0| = |Y| + 2|M|$, and $|X| = |Y| + |T \cap S|$, a computation gives $|M_1| = |V| - |S| - |T| + |M| + |X| = |V| - |S| - |T| + \nu(G, D; S, T)$. This gives the required inequality.

Let M_V denote the matching $\{v\hat{v} : v \in V - (S \cup T)\}$ of H. Let M_H be a matching of H of size $\nu(H)$ for which $|M_H \cap M_V|$ is as large as possible. Let $\widehat{X} = (\widehat{S} \setminus \widehat{T}) \cap V(M_H)$, $T_0 = (T \setminus S) \cap V(M_H)$ and let $M_G = M_H \cap E(G)$. Consider the graph $H' = (V(H), (M_H - M_G) \cup M_V)$. The components of H' are either edges of $M_H \cap M_V$, or paths or even cycles in which edges alternate between $M_H - M_G$ and M_V, and in which vertices alternate between V and $\widehat{V}$. The set of isolated vertices of H is $(T \setminus T_0) \cup (\widehat{S} \setminus \widehat{X})$. Each vertex in $\widehat{X} \cup T_0 \cup V(M_G)$ has degree 1 in H and each vertex in $V(H) \setminus (\widehat{S} \cup T)$ has degree at least 1 in H.

If Q is a path component of H with an end edge $u\hat{u} \in M_V \setminus M_H$ and $u \notin V(M_G)$, then the corresponding end vertex of Q is unmatched in M_H, so $M_H \Delta E(Q)$ is a matching of size at least $\nu(H)$ containing more edges of M_V than M_H does, a contradiction. Therefore $u \in V(M_G)$ for every such component. Moreover, every $u \in V(M_G)$ is contained in a path component of this sort. Combining the above information with the alternating conditions on edges and vertices, it follows that every component of H is either

(a) an isolated vertex in $((\widehat{S} \setminus \widehat{T}) \setminus \widehat{X}) \cup ((T \setminus S) \setminus T_0)$,
(b) an edge in $M_V \cup M_H$,
(c) an even cycle contained in $V(H) \setminus (\widehat{S} \cup T_0)$, or
(d) a path with one end in $\widehat{X} \cup V(M_G)$, another end in T_0, and no internal vertex in $\widehat{S} \cup T$.

Therefore $V(M_H) = V(H) - ((\widehat{S} \setminus \widehat{T}) \setminus \widehat{X}) \cup ((T \setminus S) \setminus T_0)$. Moreover, the set of M_H-edges in each path Q of type (d) corresponds to a dipath P in D from the end of Q in $X \cup V(M_G)$ to the end of Q in T_0, so the set of paths of type (d) together imply that $X \cup V(M_G)$ is T-linked in D and $|T_0| = 2|M_G| + |X|$. Thus $\nu(H) = \frac{1}{2}(|V(H)| - |S \setminus T| + |X| - |T \setminus S| + |T_0|)$. Using $|V(H)| = 2|V| - |T| - |S|$ and $|T_0| = 2|M_G| + |X|$, we get $\nu(H) = |V| - |T| - |S| + |S \cap T| + |M_G| + |X|$. But $M_G \cup X$ is T-linked in D and so is $(M_G \cup X) \cup (S \cap T)$ by adding trivial paths, so $\nu(G, D; S, T) \geq |M_G| + |X| + |S \cap T|$ and the lemma follows. □

We now prove Theorem 1.2, rephrasing it as a 'min-max' theorem.

Theorem 3.2. *Let $G = (V, E)$ be a graph, $D = (V, A)$ be a digraph and S, T be subsets of V. Then*

$$\nu(G, D; S, T) = \min_{S', T', \mathcal{P}} \sum_{P \in \mathcal{P}} \left\lfloor \tfrac{1}{2} \left(|P \cap S'| + |P \cap T'| \right) \right\rfloor$$

where the minimum is taken over all $S \subseteq S' \subseteq V$, $T \subseteq T' \subseteq V$ and partitions $\mathcal{P}$ of V that are crossed by no edge of $G - S'$ or arc of $D^{S', T'}$.

Proof. We first show that for any $S', T', \mathcal{P}$ chosen as above, the summation in the formula is an upper bound for $\nu(G, D; S, T)$. Since $\nu(G, D; S, T) \leq \nu(G, D; S', T')$ whenever $S \subseteq S'$, $T \subseteq T'$, it suffices to assume that $S' = S$ and $T' = T$. Let $\mathcal{P}$ be a partition of V crossed by no edge of $G - S$ or arc of $D^{S,T}$. If $\nu(G, D; S, T) = k$, then there is a matching M of G and a set $X \subseteq S \setminus V(M)$ with $|X| + |M| = k$ and an $(X \cup V(M), T)$-linkage $\mathcal{Q}$ in D. It is clear that we can choose X, M and $\mathcal{Q}$ so that $S \cap V(M) = \varnothing$ and so that no dipath in $\mathcal{Q}$ has an internal vertex in $S \cup T$; therefore each path in $\mathcal{Q}$ and edge in M is contained in a block of $\mathcal{P}$. Let $T_0 \subseteq T$ be the set of final vertices of paths in $\mathcal{Q}$, so $|T_0| = |X| + 2|M|$. Each edge of M contributes two vertices of T_0 to its block and each vertex in X contributes one vertex of each of X and T_0 to its block, so for each $P \in \mathcal{P}$ the quantity $|P \cap X| + |P \cap T_0|$ is even and thus at most $2 \left\lfloor \frac{1}{2} \left(|P \cap S| + |P \cap T| \right) \right\rfloor$. Summing over all $P \in \mathcal{P}$, we see that $2k = 2(|X| + |M|) = |X| + |T_0| \leq 2 \sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2} \left(|P \cap S| + |P \cap T| \right) \right\rfloor$, as required.

It now suffices to show that there exists a partition $\mathcal{P}$ where equality holds. Let $H = H(G, D; S, T)$. By Theorem 2.1, there is a set $Z \subseteq V(H)$ such that $\nu(H) = |Z| + \sum_C \left\lfloor \frac{1}{2}|V(C)| \right\rfloor$, where we sum over components C of $H - Z$. Let $Z = U \cup \widehat{W}$ and let $\mathcal{C}$ denote the set of components of $H - Z$. For each $C \in \mathcal{C}$ let $P(C) = P_1 \cup P_2$, where $V(C) = \widehat{P_1} \cup P_2$. Let $\mathcal{P}' = \{P(C) : C \in \mathcal{C}\}$ and $\mathcal{P} = \mathcal{P}' \cup \{\{v\} : v \in S' \cap T'\}$, noting that $\mathcal{P}'$ is a partition of $V \setminus (S' \cup T')$ and $\mathcal{P}$ is a partition of V. By construction of H, no edge of $G - U$ or arc of $D^{U,W}$ crosses $\mathcal{P}$.

Let $S' = S \cup U$, $T' = T \cup W$. The vertices $v \in V$ for which $\{v, \widehat{v}\} \subseteq V(H - Z)$ are exactly those in $V \setminus (S' \cup T')$, and each such pair $v, \widehat{v}$ is joined by an edge of $H - Z$. For each $C \in \mathcal{C}$ with $V(C) = \widehat{P_1} \cup P_2$, we therefore have $P_2 \cap (V \setminus (S' \cup T')) = P_1 \cap (V \setminus (S' \cup T'))$, so

$$\begin{aligned}\left\lfloor \tfrac{1}{2}|V(C)| \right\rfloor &= \left\lfloor \tfrac{1}{2}\left(|P_1| + |P_2|\right) \right\rfloor \\ &= \left\lfloor \tfrac{1}{2}\left(|P_1 \cap T'| + |P_2 \cap S'| + 2|P_1 \cap (V \setminus (S' \cup T'))|\right) \right\rfloor \\ &= \left\lfloor \tfrac{1}{2}\left(|P(C) \cap T'| + |P(C) \cap S'|\right) \right\rfloor + |P_1 \cap (V \setminus (S' \cup T'))|,\end{aligned}$$

since $P_2 \cap T' = P_1 \cap S' = \varnothing$. Summing over all $C \in \mathcal{C}$ gives

$$\begin{aligned}\sum_{C \in \mathcal{C}} \left\lfloor \tfrac{1}{2}|V(C)| \right\rfloor &= |V \setminus (S' \cup T')| + \sum_{C \in \mathcal{C}} \left\lfloor \tfrac{1}{2}\left(|P(C) \cap T'| + |P(C) \cap S'|\right) \right\rfloor \\ &= |V \setminus (S' \cup T')| + \sum_{P \in \mathcal{P}'} \left\lfloor \tfrac{1}{2}\left(|P \cap S'| + |P \cap T'|\right) \right\rfloor .\end{aligned}$$

Every block in $\mathcal{P} \setminus \mathcal{P}'$ is a singleton in $S' \cap T'$, so $\sum_{P \in \mathcal{P} \setminus \mathcal{P}'} \left\lfloor \frac{1}{2}\left(|P \cap S'| + |P \cap T'|\right) \right\rfloor = |S' \cap T'|$. With the above this gives $\sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}|V(C)| \right\rfloor = \sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2}\left(|P \cap S'| + |P \cap T'|\right) \right\rfloor + |V| - |S'| - |T'|$. The required equality now follows from definition of S' and T', Lemma 3.1 and the fact that $\nu(H) = |U| + |W| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}|V(C)| \right\rfloor$. □

4. Applications

We saw earlier that setting $U = \varnothing$ and $T = V$ in Theorem 1.2 yields the Tutte-Berge formula; another special case gives Menger's theorem for vertex-disjoint paths in a digraph:

Theorem 4.1. *Let $D = (V, A)$ be a digraph and S and T be subsets of V. Either there are k vertex-disjoint dipaths from S to T in D or there is a set $X \subseteq V$ so that $|X| < k$ and there are no dipaths from S to T in $D - X$:*

Proof. We set $G = (V, \varnothing)$ and apply Theorem 1.2. If there are no k vertex-disjoint (S, T)-dipaths in D then there are sets $S' \supseteq S$, $T' \supseteq T$ and a partition $\mathcal{P}$ of V crossed by no edges of $D^{S',T'}$ so that $\sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2} (|S' \cap P| + |T' \cap P|) \right\rfloor < k$. Note that each minimal (S', T')-dipath in D is contained in a block of $\mathcal{P}$. For each $P \in \mathcal{P}$ let $X_P = P \cup S'$ if $|P \cup S'| \le |P \cup T'|$ and $X_P = P \cup T'$ otherwise. Let $X = \cup_{P \in \mathcal{P}} X_P$. Now $|X_P| \le \left\lfloor \frac{1}{2} (|S' \cap P| + |T' \cap P|) \right\rfloor$ for each $P \in \mathcal{P}$ so $|X| < k$. But by construction, no block of $\mathcal{P}$ contains both a vertex of $S' \setminus X$ and a vertex of $T' \setminus X$, so there are no (S', T')-dipaths in $D - X$, giving the result. □

Our next corollary is a 'qualitative' version of Theorem 1.2 with a cleaner statement.

Theorem 4.2. *Let G be a graph and D be a digraph with common vertex set V, let $T \subseteq V$ and let k be a positive integer. Either G has a k-edge matching whose vertex set is T-linked in D, or there is a set $X \subseteq V$ with $|X| \le 2k - 2$ such that $\{u, v\}$ is not T-linked in $D - X$ for any edge uv of G.*

Proof. If G has no such matching, then by Theorem 1.1 there are sets $S' \subseteq V$, $T \subseteq T' \subseteq V$ and a partition $\mathcal{P}$ of V crossed by no edge of $G - S'$ or arc of $D^{S',T'}$ so that $\sum_{P \in \mathcal{P}} \left\lfloor \frac{1}{2} (|S' \cap P| + |T' \cap P|) \right\rfloor \le k - 1$. Let X be a set formed by choosing all but one element of $|(S' \cup T') \cap P|$ from each $P \in \mathcal{P}$; note that $|X| \le 2(k-1)$. It is clear that no vertex in S' is T-linked in $D - X$, and if $uv \in E(G - S')$ then any minimal $(\{u, v\}, T)$-linkage in $D - X$ is contained in a block of $\mathcal{P}$ so cannot exist by choice of X. Therefore there is no edge of G whose set of ends is T-linked in $D - X$, as required. □

If S and T are sets of vertices in a digraph D, then we say that S is *doubly T-linked* in D if there are disjoint (S, T)-linkages $\mathcal{P}_1$ and $\mathcal{P}_2$ in D such that the $2|S|$ dipaths in $\mathcal{P}_1 \cup \mathcal{P}_2$ have only initial vertices in common. Our final corollary gives a qualitative obstruction to large doubly T-linked sets.

Theorem 4.3. *Let $D = (V, A)$ be a digraph and $S, T \subseteq V$. Either there exists a doubly T-linked k-element subset of S or there is a set $Z \subseteq V$ such that $|Z| \le 2k - 2$ and there is no $x \in S$ for which $\{x\}$ is doubly T-linked in $D - Z$.*

Proof. Let $\widehat{S} = \{\hat{s} : s \in S\}$ be a copy of S disjoint from V and let $V^+ = V \cup \widehat{S}$. Let $G^+ = (V^+, \{s\hat{s} : s \in S\})$ and $D^+ = (V^+, A \cup \{(\hat{s}, v) : s \in S, (s, v) \in A\})$ (the copies of vertices in S therefore have no in-neighbours in D^+). Note that a set $S_0 \subseteq S$ is doubly T-linked in D if and only if the vertex set of the corresponding matching $\{s\hat{s} : s \in S_0\}$ of G^+ is T-linked in D^+. If there is no doubly T-linked k-element subset of S in D, then by Theorem 4.2 there is a set $Z \subseteq V^+$ such that $|Z| \leq 2k-2$ and for all $s \in S$ the set $\{s, \hat{s}\}$ is not T-linked in $D^+ - Z$. It is clear that Z can be chosen to contain no vertices of $\widehat{S}$, and therefore that Z satisfies the theorem. □

In the special case of graphs (in other words, when $(u, v) \in A$ if and only if $(v, u) \in A$), the above is equivalent to Theorem 2.1 of [1].

5. Acknowledgements

We thank Jim Geelen for suggesting the problem and for his useful advice.

6. References

[1] N. Kakimura, K. Kawarabayashi, D. Marx, Packing cycles through prescribed vertices, J. Comb. Theory, Ser. B 101 (2011), 378–381.
[2] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer, Algorithms and Combinatorics 23 (2003).
[3] P. Tong, E.L. Lawler, V.V. Vazirani, Solving the weighted parity problem for gammoids by reduction to graphic matching, Technical Report UCB/CSD-82-103, EECS Department, University of California, Berkeley (1982).

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