ON MINOR-CLOSED CLASSES OF MATROIDS WITH EXPONENTIAL GROWTH RATE

JIM GELEEN AND PETER NELSON

Abstract. Let $M$ be a minor-closed class of matroids that does not contain arbitrarily long lines. The growth rate function, $h : \mathbb{N} \to \mathbb{N}$ of $M$ is given by

$$h(n) = \max \{|M| : M \in M \text{ is simple, and } r(M) \leq n\}.$$

The Growth Rate Theorem shows that there is an integer $c$ such that either:

1. $h(n) \leq cn$ for all $n \geq 0$, or

2. $(n+1)/2 \leq h(n) \leq cn^2$, or there is a prime-power $q$ such that $q^{n-1}/q-1 \leq h(n) \leq cq^n$; this separates classes into those of linear density, quadratic density, and base-$q$ exponential density. For classes of base-$q$ exponential density that contain no $(q^2+1)$-point line, we prove that $h(n) = q^{n-1}/q-1$ for all sufficiently large $n$. We also prove that, for classes of base-$q$ exponential density that contain no $(q^2+q+1)$-point line, there exists $k \in \mathbb{N}$ such that $h(n) = q^{n+k-1}/q-1 - q^{2k-1}/q-1$ for all sufficiently large $n$.

1. Introduction

We prove a refinement of the Growth Rate Theorem for certain exponentially dense classes. We call a class of matroids minor-closed if it is closed under both minors and isomorphism. The growth rate function, $h_M : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ for a class $M$ of matroids is defined by

$$h_M(n) = \max\{|M| : M \in M \text{ is simple, and } r(M) \leq n\}.$$

The following striking theorem summarizes the results of several papers, [1,2,4].

**Theorem 1.1** (Growth Rate Theorem). Let $M$ be a minor-closed class of matroids, not containing all simple rank-2 matroids. Then there is an integer $c$ such that either:

1. $h_M(n) \leq cn$ for all $n \geq 0$, or

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(2) \( \binom{n+1}{2} \leq h_M(n) \leq cn^2 \) for all \( n \geq 0 \), and \( M \) contains all graphic matroids, or

(3) there is a prime power \( q \) such that \( \frac{q^n - 1}{q-1} \leq h_M(n) \leq cq^n \) for all \( n \geq 0 \), and \( M \) contains all \( \text{GF}(q) \)-representable matroids.

In particular, the theorem implies that \( h_M(n) \) is finite for all \( n \) if and only if \( M \) does not contain all simple rank-2 matroids. If \( M \) is a minor-closed class satisfying (3), then we say that \( M \) is \textit{base-} \( q \) \textit{exponentially dense}. Our main theorems precisely determine, for many such classes, the eventual value of the growth rate function:

**Theorem 1.2.** Let \( q \) be a prime power. If \( M \) is a base- \( q \) exponentially dense minor-closed class of matroids such that \( U_{2,q^2+2} \notin M \), then

\[
h_M(n) = \frac{q^n - 1}{q-1}
\]

for all sufficiently large \( n \).

Consider, for example, the class \( M \) of matroids with no \( U_{2,\ell+2} \)-minor, where \( \ell \geq 2 \) is an integer. By the Growth Rate Theorem, this class is base- \( q \) exponentially dense, where \( q \) is the largest prime-power not exceeding \( \ell \). Clearly \( q^2 > \ell \), so, by Theorem 1.2, \( h_M(n) = \frac{q^n - 1}{q-1} \) for all large \( n \). This special case is the main result of [3], which essentially also contains a proof of Theorem 1.2.

**Theorem 1.3.** Let \( q \) be a prime power. If \( M \) is a base- \( q \) exponentially dense minor-closed class of matroids such that \( U_{2,q^2+q+1} \notin M \), then there is an integer \( k \geq 0 \) such that

\[
h_M(n) = \frac{q^{n+k} - 1}{q-1} - q^{2k} - 1
\]

for all sufficiently large \( n \).

Consider, for example, any proper minor-closed subclass \( M \) of the \( \text{GF}(q^2) \)-representable matroids that contains all \( \text{GF}(q) \)-representable matroids. Such classes are all base- \( q \) exponentially dense and do not contain \( U_{2,q^2+2} \), so Theorem 1.3 applies; this special case is the main result of [8].

If the hypothesis of Theorem 1.3 is weakened to allow \( U_{2,q^2+q+1} \in M \), then the conclusion no longer holds. Consider the class \( M_1 \) defined to be the set of truncations of all \( \text{GF}(q) \)-representable matroids; note that \( U_{2,q^2+q+2} \notin M_1 \) and \( h_{M_1}(n) = \frac{q^{n+1} - 1}{q-1} \) for all \( n \geq 2 \).

More generally, for each \( k \geq 0 \), if \( M_k \) is the set of matroids obtained from \( \text{GF}(q) \)-representable matroids by applying \( k \) truncations, then
h_M(n) = \frac{2^{n+k-1}}{q-1} for all n ≥ 2. This expression differs from that in Theorem 1.3 by only the constant \(q^{2k-1}_{q^2-1}\). It is conjectured [8,9] that, for each k, these are the extremes in a small spectrum of possible growth rate functions:

**Conjecture 1.4.** Let \(q\) be a prime power, and \(\mathcal{M}\) be a base-\(q\) exponentially dense minor-closed class of matroids. There exist integers \(k\) and \(d\) with \(k ≥ 0\) and \(0 ≤ d ≤ q^{2k-1}_{q^2-1}\), such that \(h_M(n) = \frac{q^{n+k-1}}{q-1} - qd\) for all sufficiently large \(n\).

We conjecture further that, for every allowable \(q\), \(k\) and \(d\), there exists a minor-closed class with the above as its eventual growth rate function.

There is a stronger conjecture [9] regarding the exact structure of the extremal matroids. For a non-negative integer \(k\), a \(k\)-element projection of a matroid \(M\) is a matroid of the form \(N/C\), where \(N\setminus C = M\), and \(C\) is a \(k\)-element set of \(N\).

**Conjecture 1.5.** Let \(q\) be a prime power, and \(\mathcal{M}\) be a base-\(q\) exponentially dense minor-closed class of matroids. There exists an integer \(k ≥ 0\) such that, if \(M ∈ \mathcal{M}\) is a simple matroid of sufficiently large rank with \(|M| = h_M(r(M))\), then \(M\) is the simplification of a \(k\)-element projection of a projective geometry over \(\text{GF}(q)\).

We will show, as was observed in [9], that this conjecture implies the previous one; see Lemma 3.1.

2. Preliminaries

A matroid \(M\) is called \((q, k)\)-full if

\[ ε(M) ≥ \frac{q^{r(M)+k-1}}{q-1} - q^{2k-1}_{q^2-1}; \]

moreover, if strict inequality holds, \(M\) is \((q, k)\)-overfull.

Our proof of Theorem 1.3 follows a strategy similar to that in [8]; we show that, for any integer \(n > 0\), every \((q, k)\)-overfull matroid in \(\text{EX}(U_2,q^2,q+1)\), with sufficiently large rank, contains a \((q, k + 1)\)-full rank-\(n\) minor. The Growth Rate Theorem tells us that a given base-\(q\) exponentially dense minor-closed class cannot contain \((q, k)\)-full matroids for arbitrarily large \(k\), so this gives the result. Theorem 1.2 is easier and will follow along the way.

We follow the notation of Oxley [10]; flats of rank 1, 2 and 3 are respectively points, lines and planes of a matroid. If \(M\) is a matroid, and \(X, Y ⊆ E(M)\), then \(\cap_M(X, Y) = r_M(X) + r_M(Y) − r_M(X \cup Y)\)
is the local connectivity between $X$ and $Y$. If $\cap_M(X,Y) = 0$, then $X$ and $Y$ are skew in $M$, and if $\mathcal{X}$ is a collection of sets in $M$ such that each $X \in \mathcal{X}$ is skew to the union of the sets in $\mathcal{X} - \{X\}$, then $\mathcal{X}$ is a mutually skew collection of sets. A pair $(F_1, F_2)$ of flats in $M$ is modular if $\cap_M(F_1, F_2) = r_M(F_1 \cap F_2)$, and a flat $F$ of $M$ is modular if, for each flat $F'$ of $M$, the pair $(F, F')$ is modular. In a projective geometry each pair of flats is modular and, hence, each flat is modular.

For a matroid $M$, we write $|M|$ for $|E(M)|$, and $\varepsilon(M)$ for $|\text{si}(M)|$, the number of points in $M$. Thus, $h_M(n) = \max(\varepsilon(M) : M \in \mathcal{M}, r(M) \leq n)$. Two matroids are equal up to simplification if their simplifications are isomorphic. We let $EX(M)$ denote the set of matroids with no $M$-minor; Theorems 1.2 and 1.3 apply to subclasses of $EX(U_{2,q^2+1})$ and $EX(U_{2,q^2+q+1})$ respectively. The following theorem of Kung [5] bounds the density of a matroid in $EX(U_{2,\ell+2})$:

**Theorem 2.1.** Let $\ell \geq 2$ be an integer. If $M \in EX(U_{2,\ell+2})$, then $\varepsilon(M) \leq \frac{\ell^{r(M)} - 1}{\ell - 1}$.

The next result is an easy application of the Growth Rate Theorem.

**Lemma 2.2.** There is a real-valued function $\alpha_{2,2}(n, \beta, \ell)$ so that, for any integers $n \geq 1$ and $\ell \geq 2$, and real number $\beta > 1$, if $M \in EX(U_{2,\ell+2})$ is a matroid such that $\varepsilon(M) \geq \alpha_{2,2}(n, \beta, \ell)\beta^{r(M)}$, then $M$ has a $PG(n - 1, q)$-minor for some $q > \beta$.

The following lemma was proved in [8]:

**Lemma 2.3.** Let $\lambda, \mu$ be real numbers with $\lambda > 0$ and $\mu > 1$, let $t \geq 0$ and $\ell \geq 2$ be integers, and let $A$ and $B$ be disjoint sets of elements in a matroid $M \in EX(U_{2,\ell+2})$ with $r_M(B) \leq t < r(M)$ and $\varepsilon(M|A) > \lambda \mu^{t\lambda(A)}$. Then there is a set $A' \subseteq A$ that is skew to $B$ and satisfies $\varepsilon(M|A') > \lambda (\frac{\mu - 1}{t})^t \mu^{t\lambda(A')}$.  

3. **Projections**

Recall that a $k$-element projection of a matroid $M$ is a matroid of the form $N/C$, where $C$ is a $k$-element set of a matroid $N$ satisfying $N \setminus C = M$.

In this section we are concerned with projections of projective geometries. Consider a $k$-element set $C$ in a matroid $N$ such that $N \setminus C = PG(n + k - 1, q)$ and let $M = N/C$. Thus $M$ is a $k$-element projection of $PG(n + k - 1, q)$. Below are easy observations that we use freely.

- If $C$ is not independent, then $M$ is a $(k - 1)$-element projection of $PG(n + k - 1, q)$.  

• If $C$ is not coindependent, then $M$ is a $(k-1)$-element projection of $\text{PG}(n + k - 1, q)$.

• If $C$ is not closed in $N$, then $M$ is, up to simplification, a $(k-1)$-element projection of $\text{PG}(n + k - 2, q)$.

• $M$ has a $\text{PG}(r(M) - 1, q)$-restriction.

Our next result gives the density of projections of projective geometries; given such a projection $M$, this density is determined to within a small range by the minimum $k$ for which $M$ is a $k$-element projection. As mentioned earlier, this lemma also tells us that Conjecture 1.5 implies Conjecture 1.4.

**Lemma 3.1.** Let $q$ be a prime power, and $k \geq 0$ be an integer. If $N$ is a matroid, and $C$ is a rank-$k$ flat of $N$ such that $N \setminus C \cong \text{PG}(r(N) - 1, q)$, then $\varepsilon(N/C) = \varepsilon(N \setminus C) - qd$ for some $d \in \{0, 1, \ldots, \frac{q^{2k-1}}{q^2-1}\}$.

**Proof.** Each point $P$ of $N/C$ is a flat of the projective geometry $N \setminus C$, so $|P| = \frac{q^{N(P)-1}}{q-1} = 1 + q\frac{q^{N(P)-1}-1}{q-1}$. Therefore $\varepsilon(N/C) - \varepsilon(N/C)$ is a multiple of $q$.

Let $\mathcal{P}$ denote the set of all points in $N/C$ that contain more than one element, and let $F$ be the flat of $N \setminus C$ spanned by the union of these points. Choose a minimal set $\mathcal{P}_0 \subseteq \mathcal{P}$ of points spanning $F$ in $N/C$ (so $|\mathcal{P}_0| = r_{N/C}(F)$); if possible choose $\mathcal{P}_0$ so that it contains a set in $P \in \mathcal{P}$ with $r_{N}(P) > 2$. Note that: (1) the points in $\mathcal{P}_0$ are mutually skew in $N/C$, (2) each pair of flats of $N \setminus C$ is modular, and (3) $C$ is a flat of $N$. It follows that $\mathcal{P}_0$ is a mutually skew collection of flats in $N \setminus C$. Now, for each $P \in \mathcal{P}_0$, $r_{N}(P) > r_{N/C}(P)$. Therefore, since $r(N) - r(N/C) = k$, we have $r_{N/C}(F) = |\mathcal{P}_0| \leq k$. Moreover, if $r_{N/C}(F) = k$, then each set in $\mathcal{P}_0$ is a line of $N \setminus C$, and, hence, by our choice of $\mathcal{P}_0$, each set in $\mathcal{P}$ is a line in $N \setminus C$.

If $r_{N/C}(F) = k$, then we have $|F| = \frac{q^{2k-1}}{q-1}$ and $|\mathcal{P}| \leq |F|$. This gives $\varepsilon(N \setminus C) - \varepsilon(N/C) \leq q\frac{|F|}{q+1} = q\frac{q^{2k-1}}{q^2-1}$, as required.

If $r_{N/C}(F) < k$, then $\varepsilon(N \setminus C) - \varepsilon(N/C) \leq |F| \leq \frac{q^{2k-1}}{q-1}$. It is routine to verify that $\frac{q^{2k-1}}{q-1} < q\frac{q^{2k-1}}{q^2-1}$, which proves the result. \hfill $\square$

The next two lemmas consider single-element projections, highlighting the importance of $U_{2,q^2+1}$ and $U_{2,q^2+q+1}$ in Theorems 1.2 and 1.3.

**Lemma 3.2.** Let $q$ be a prime power and let $e$ be an element of a matroid $M$ such that $M \setminus e \cong \text{PG}(r(M) - 1, q)$. Then there is a unique minimal flat $F$ of $M \setminus e$ that spans $e$. Moreover, if $r(M) \geq 3$ and $r_M(F) \geq 2$, then $M/e$ contains a $U_{2,q^2+1}$-minor, and if $r_M(F) \geq 3$, then $M/e$ contains a $U_{2,q^2+q+1}$-minor.
Proof. If $F_1$ and $F_2$ are two flats of $M \setminus e$ that span $e$, then, since $r_M(F_1 \cap F_2) + r_M(F_1 \cup F_2) = r_M(F_1) + r_M(F_2)$, it follows that $F_1 \cap F_2$ also spans $e$. Therefore there is a unique minimal flat $F$ of $M \setminus e$ that spans $e$. The uniqueness of $F$ implies that $e$ is freely placed in $F$.

Suppose that $r_M(F) \geq 3$. Thus $(M/e)\setminus F$ is the truncation of a projective geometry of rank $\geq 3$. So $M/e$ contains a truncation of $\PG(2, q)$ as a minor; therefore $M/e$ has a $U_{2,q^2+q+1}$-minor.

Now suppose that $r(M) \geq 3$ and that $r_M(F) = 2$. If $F'$ is a rank-$3$ flat of $M \setminus e$ containing $F$, then $\varepsilon((M/e)\setminus F') = q^2 + 1$, so $M/e$ has a $U_{2,q^2+1}$-minor. □

An important consequence is that, if $M$ is a simple matroid with a $\PG(r(M) - 1, q)$-restriction $R$ and no $U_{2,q^2+q+1}$-minor, then every $e \in E(M) - E(R)$ is spanned by a unique line of $R$. The next result describes the structure of the projections in $\operatorname{EX}(U_{2,q^2+q+1})$.

**Lemma 3.3.** Let $q$ be a prime power, and $M \in \operatorname{EX}(U_{2,q^2+q+1})$ be a simple matroid, and $e \in E(M)$ be such that $M \setminus e \cong \PG(r(M) - 1, q)$. If $L$ is the unique line of $M \setminus e$ that spans $e$, then $L$ is a point of $M/e$, and each line of $M/e$ containing $L$ has $q^2 + 1$ points and is modular.

**Proof.** Let $L'$ be a line of $M/e$ containing $L$. Then $L'$ is a plane of $M \setminus e$, so, by Lemma 3.2, $L'$ has $q^2 + 1$ points in $M/e$.

Note that $e$ is freely placed on the line $L \cup \{e\}$ in $M$. It follows that $M$ is $\operatorname{GF}(q^2)$-representable. Now $L'$ is a $(q^2 + 1)$-point line in the $\operatorname{GF}(q^2)$-representable matroid $M/e$; hence, $L'$ is modular in $M/e$. □

4. **Dealing with long lines**

This section contains two lemmas that construct a $U_{2,q^2+q+1}$-minor of a matroid $M$ with a $\PG(r(M) - 1, q)$-restriction $R$ and some additional structure.

**Lemma 4.1.** Let $q$ be a prime power, and $M$ be a simple matroid of rank at least $7$ such that

- $M$ has a $\PG(r(M) - 1, q)$-restriction $R$, and
- $M$ has a line $L$ containing at least $q^2 + 2$ points, and
- $E(M) \neq E(R) \cup L$,

then $M$ has a $U_{2,q^2+q+1}$-minor.

**Proof.** We may assume that $E(M) = E(R) \cup L \cup \{z\}$, where $z \notin L \cup E(R)$. Let $F$ be a minimal flat of $R$ that spans $L \cup \{z\}$. It follows easily from Lemma 3.2, that either $M$ has a $U_{2,q^2+q+1}$-minor or $r_M(F) \leq 6$. To simplify the proof we will prove the lemma with the weaker hypothesis that $r(M) \geq 1 + r_M(F)$, in place of the hypothesis
that \(r(M) \geq 7\), and we will suppose that \((M, R, L)\) forms a minimum rank counterexample under these weakened hypotheses.

Let \(L_z\) denote the line of \(R\) that spans \(z\) in \(M\). Since \(z \not\in L\), we have \(r_M(L \cup L_z) \geq 3\). We may assume that \(r_M(L \cup L_z) = 3\), since otherwise we could contract a point in \(F - (L \cup L_z)\) to obtain a smaller counterexample. Similarly, we may assume that \(r_M(F) = 3\) and \(r(M) = 4\), as otherwise we could contract an element of \(F - cl_M(L \cup L_z)\) or \(E(M) - cl_M(F)\).

By Lemma 3.3, \(L_z\) is a point of \((M/z)\)[R] and each line of \((M/z)\)[R] is modular and has \(q^2 + 1\) points. One of these lines is \(F\), and, since \(F\) spans \(L\), \(F\) spans a line with \(q^2 + 2\) points in \(M/z\). Let \(e \in cl_{M/z}(F)\) be an element that is not in parallel with any element of \(F\). Since \(F\) is a modular line in \((M/z)\)[R], the point \(e\) is freely placed on the line \(F \cup \{e\}\) in \((M/z)[(R \cup \{e\})]. Therefore \(\varepsilon(M/\{e, z\}) = \varepsilon((M/\{z\})[R] - q^2 = 1 + q^2(q + 1) - q^2 = q^3 - 1\), contradicting the fact that \(M \in EX(U_{2,q^2+q+1}).\)

**Lemma 4.2.** Let \(q\) be a prime power, and \(k \geq 3\) be an integer. If \(M\) is a matroid of rank at least \(k + 7\), with a \(PG(r(M) - 1, q)\)-restriction, and a set \(X \subseteq E(M)\) with \(r_M(X) \leq k\) and \(\varepsilon(M|X) > \frac{q^{2k} - 1}{q^2 - 1}\), then \(M\) has a \(U_{2,q^2+q+1}\)-minor.

**Proof.** Let \(M_0\) be a matroid satisfying the hypotheses, with a \(PG(r(M_0) - 1, q)\)-restriction \(R_0\). We may assume that \(M_0 \in EX(U_{2,q^2+q+1})\), and by choosing a rank-\(k\) set containing \(X\), we may also assume that \(r_{M_0}(X) = k\). By Lemma 3.2, \(R_0\) has a flat \(F_0\) of rank at most \(2k\) such that \(X \subseteq cl_{M_0}(F_0)\). By contracting at most \(k\) points in \(F_0 - cl_{M_0}(X)\), we obtain a minor \(M\) of \(M_0\), of rank at least \(7\), such that \(r_M(X) = k\), and \(M\) has a \(PG(r(M) - 1, q)\)-restriction \(R\), and there is a rank-\(k\) flat \(F\) of \(R\) such that \(X \subseteq cl_M(F)\).

We may assume that \(M\) is simple and that \(X\) is a flat of \(M\), so \(F \subseteq X\). Let \(n = |F| = \frac{q^k - 1}{q^2 - 1}\). By Lemma 3.2, each point of \(X\) is spanned in \(M\) by a line of \(R|F\). There are \(\binom{q}{2}/\binom{q + 1}{2}\) such lines, each containing \(q + 1\) points of \(F\). If each of these lines spans at most \((q^2 - q)\) points of \(X - F\), then

\[
|X| = |F| + |X - F| \leq \frac{q^k - 1}{q - 1} + \frac{(q^2 - q)(q + 1)}{\binom{q + 1}{2}} = \frac{q^{2k} - 1}{q^2 - 1},
\]

contradicting the definition of \(X\). Therefore, some line \(L\) of \(M|X\) contains at least \(q^2 + 2\) points. We also have \(|L| \leq q^2 + q\), so a calculation gives \(|X - L| > \frac{q^{2k} - 1}{q^2 - 1} - (q^2 + q) \geq \frac{q^{k - 1}}{q - 1} = |F|\), so \(X \neq F \cup L\). Applying Lemma 4.1 to \(M|(E(R) \cup X)\) gives the result. \(\square\)
5. Matchings and unstable sets

For an integer $k \geq 0$, a \emph{$k$-matching} of a matroid $M$ is a mutually skew $k$-set of lines of $M$. Our first theorem was proved in [8], and also follows routinely from the much more general linear matroid matching theorem of Lovász [7]:

\textbf{Theorem 5.1.} There is an integer-valued function $f_{5.1}(q, k)$ so that, for any prime power $q$ and integers $n \geq 1$ and $k \geq 0$, if $\mathcal{L}$ is a set of lines in a matroid $M \cong PG(n-1, q)$, then either

(i) $\mathcal{L}$ contains a $(k+1)$-matching of $M$, or

(ii) there is a flat $F$ of $M$ with $r_M(F) \leq k$, and a set $\mathcal{L}_0 \subseteq \mathcal{L}$ with $|\mathcal{L}_0| \leq f_{5.1}(q, k)$, such that every line $L \in \mathcal{L}$ either intersects $F$, or is in $\mathcal{L}_0$. Moreover, if $r_M(F) = k$, then $\mathcal{L}_0 = \emptyset$.

We now define a property in terms of a matching in a spanning projective geometry. Let $q$ be a prime power, $M \in EX(U_{2,q^2+q+1})$ be a simple matroid with a $PG(r(M) - 1, q)$-restriction $R$, and $X \subseteq E(M \setminus R)$ be a set such that $M|(E(R) \cup X)$ is simple. Recall that, by Lemma 3.2, each $x \in X$ lies in the closure of exactly one line $L_x$ of $R$. We say that $X$ is $R$-\emph{unstable} in $M$ if the lines $\{L_x : x \in X\}$ are a matching of size $|X|$ in $R$.

\textbf{Lemma 5.2.} There is an integer-valued function $f_{5.2}(q, k)$ so that, for any prime power $q$ and integer $k \geq 0$, if $M \in EX(U_{2,q^2+q+1})$ is a matroid of rank at least 3 with a $PG(r(M) - 1, q)$-restriction $R$, then either

(i) there is an $R$-unstable set of size $k + 1$ in $M$, or

(ii) $R$ has a flat $F$ with rank at most $k$ such that $\varepsilon(M/F) \leq \varepsilon(R/F) + f_{5.2}(q, k)$.

\textbf{Proof.} Let $q$ be a prime power, and $k \geq 0$ be an integer. Set $f_{5.2}(q, k) = (q^2 + q)f_{5.1}(q, k)$. Let $M$ be a matroid with a $PG(r(M) - 1, q)$-restriction $R$. We may assume that $M$ is simple, and that the first outcome does not hold. Let $\mathcal{L}$ be the set of lines $L$ of $R$ such that $|cl_M(L)| > |cl_R(L)|$. If $\mathcal{L}$ contains a $(k+1)$-matching of $R$, then choosing a point from $cl_M(L) - cl_R(L)$ for each line $L$ in the matching gives an $R$-unstable set of size $k + 1$. We may therefore assume that $\mathcal{L}$ contains no such matching. Thus, let $F$ and $\mathcal{L}_0$ be the sets defined in the second outcome of Theorem 5.1. Let $D = \cup_{L \in \mathcal{L}_0} cl_M(L)$. We have $|D| \leq (q^2 + q)|\mathcal{L}_0| \leq f_{5.2}(q, k)$. By Lemma 3.2, each element of $M \setminus D$ either lies the closure of a line in $\mathcal{L}$ or in a point of $R$, so is parallel in $M/F$ to an element of $R$. Therefore, $\varepsilon(M/F) \leq \varepsilon(R/F) + |D|;$ the result now follows. \qed
We use an unstable set to construct a dense minor. Recall that
$(q, k)$-full and $(q, k)$-overfull were defined at the start of Section 2.

**Lemma 5.3.** Let $q$ be a prime power, and $k \geq 1$ and $n > k$ be integers. If $M \in \text{EX}(U_{2,q^2+q+1})$ is a matroid of rank at least $n + k$ with a $\text{PG}(r(M) - 1, q)$-restriction $R$, and $X$ is an $R$-unstable set of size $k$ in $M$, then $M$ has a rank-$n$ $(q, k)$-full minor $N$ with a $U_{2,q^2+1}$-restriction.

**Proof.** We may assume by taking a restriction if necessary that $r(M) = n + k$, and $E(M) = E(R) \cup X$; we show that $N = M/X$ has the required properties. For each $x \in X$, let $L_x$ denote the line of $R$ that spans $X$; thus $\{L_x : x \in X\}$ is a matching. By the definition of instability, it is clear that $X$ is independent, so $r(N) = n$. Let $x \in X$, and $P$ be a plane of $R$ that contains $L_x$ and is skew to $X - \{x\}$. By Lemma 3.3, $(M/x)|P$ has a $U_{2,q^2+1}$-restriction. Since $X - \{x\}$ is skew to $P$, $M/X$ also has a $U_{2,q^2+1}$-restriction.

To complete the proof it is enough, by Lemma 3.1, to show that $\text{cl}_M(X)$ is disjoint from $R$. This is trivial if $X$ is empty, so consider $x \in X$ and let $R' = \text{si}(R/L_x)$. Note that $R' \cong \text{PG}(n + k - 3, q)$ is a spanning restriction of $M/L_x$ and $X - \{x\}$ is $R'$-unstable. Inductively, we may assume that $\text{cl}_M/L_x(X - \{x\})$ is disjoint from $R/L_x$, but this implies that $\text{cl}_M(X)$ is disjoint from $R$, as required. \(\square\)

6. The spanning case

In this section we consider matroids that are spanned by a projective geometry.

**Lemma 6.1.** There is an integer-valued function $f_{6,1}(n, q, k)$ such that, for any prime power $q$ and integers $k \geq 0$ and $n > k + 1$, if $M \in \text{EX}(U_{2,q^2+q+1})$ is a matroid of rank at least $f_{6,1}(n, q, k)$ such that

- $M$ has a $\text{PG}(r(M) - 1, q)$-restriction $R$, and
- $M$ is $(q, k)$-overfull,

then $M$ has a rank-$n$ $(q, k + 1)$-full minor $N$ with a $U_{2,q^2+1}$-restriction.

**Proof.** Let $k \geq 0$ and $n > k + 1$ be integers, and $q$ be a prime power. Let $m > \max(k + 7, n + k + 1)$ be an integer such that

\[
\frac{q^{r+k} - 1}{q - 1} - q^{2k} - \frac{1}{q^2 - 1} > \frac{q^{r+j} - 1}{q - 1} + \max(q^2 + q, (q^2 - q) f_{5,1}(q, k))
\]

for all $r \geq m$ and $0 \leq j < k$. We set $f_{6,1}(n, q, k) = m$.

Let $M \in \text{EX}(U_{2,q^2+q+1})$ be a $(q, k)$-overfull matroid of rank at least $m$, and let $R$ be a $\text{PG}(r(M) - 1, q)$-restriction of $M$. We will show that $M$ has the required minor $N$; we may assume that $M$ is simple.
6.1.1. If $k \geq 1$, then no line of $M$ contains more than $q^2 + 1$ points.

Proof of claim: Let $L$ be a line of $M$ containing at least $q^2 + 2$ points. We have $|L| \leq q^2 + q$, so $|E(R) \cup L| \leq \frac{q^r(M) - 1}{q - 1} + q^2 + q < |M|$ by the definition of $m$. Therefore, there is a point of $M$ in neither $R$ nor $L$. By Lemma 4.1, $M$ has a $U_{2,q^2+q+1}$-minor, a contradiction. □

Let $\mathcal{L}$ be the set of lines of $R$, and $\mathcal{L}^+$ be the set of lines of $R$ that are not lines of $M$; note that each $L \in \mathcal{L}^+$ contains exactly $q + 1$ points of $R$, and spans an extra point in $M$. By Lemma 3.2, every point of $M \setminus E(R)$ is spanned by a line in $\mathcal{L}^+$.

6.1.2. $\mathcal{L}^+$ contains a $(k + 1)$-matching of $R$.

Proof of claim: If $k = 0$, then since $|M| > |R|$, we must have $\mathcal{L}^+ \neq \emptyset$, so the claim is trivial. Thus, assume that $k \geq 1$ and that there is no such matching. Let $F \subseteq E(R)$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ be the sets defined in Theorem 5.1. Let $j = r_M(F)$; we know that $0 \leq j \leq k$, and that $\mathcal{L}_0$ is empty if $j = k$. Let $\mathcal{L}_F = \{L \in \mathcal{L} : |L \cap F| = 1\}$. By definition, every point of $M \setminus R$ is in the closure of $F$, or the closure of a line in $\mathcal{L}_F \cup \mathcal{L}_0$.

Every point of $R \setminus F$ lies on exactly $|F|$ lines in $\mathcal{L}_F$, and each such line contains exactly $q$ points of $R \setminus F$, so

$$|\mathcal{L}_F| = \frac{|F||R \setminus F|}{q} = \frac{(q^j - 1)(q^{r(M)} - q^j)}{q(q - 1)^2}.$$ 

Furthermore, each line in $\mathcal{L}$ contains $q + 1$ points of $R$, and its closure in $M$ contains at most $q^2 - q$ points of $M \setminus R$ by the first claim. We argue that $|\text{cl}_M(F)| \leq \frac{q^{2j - 1}}{q^2 - 1}$; if $j \leq 2$, then this follows from the first claim, and otherwise, we have $r(M) \geq m \geq k + 7$, so the bound follows by applying Lemma 4.2 to $M$ and $\text{cl}_M(F)$. We now estimate $|M|$.

$$|M| = |R| + |M \setminus E(R)|$$

$$\leq |R| + \sum_{L \in \mathcal{L} \setminus \mathcal{L}_0} |\text{cl}_M(L) - E(R)| + |\text{cl}_M(F) - F|$$

$$\leq \frac{q^{r(M)} - 1}{q - 1} + (q^2 - q)(|\mathcal{L}_F| + |\mathcal{L}_0|) + \left(\frac{q^{2j - 1} - q^j - 1}{q^2 - 1} - \frac{q^j - 1}{q - 1}\right).$$

Now, a calculation and our value for $\mathcal{L}_F$ obtained earlier together give $|M| \leq \frac{q^{r(M) + j - 1}}{q - 1} - q\frac{q^{2j - 1}}{q^2 - 1} + (q^2 - q)|\mathcal{L}_0|$. If $j < k$, then, since $r(M) \geq m$ and $|\mathcal{L}_0| \leq f_{5,1}(q,k)$, we have $|M| \leq \frac{q^{r(M) + k - 1}}{q - 1} - q\frac{q^{2k - 1}}{q^2 - 1}$ by definition of $m$. If $j = k$, then $|\mathcal{L}_0| = 0$, so the same inequality holds. In either case, we contradict the fact that $M$ is $(q,k)$-overfull. □
Now, $\mathcal{L}^+$ has a matching of size $k+1$, so by construction of $\mathcal{L}^+$, there is an $R$-unstable set $X$ of size $k+1$ in $M$. Since $r(M) \geq m > n+k+1$, the required minor $N$ is given by Lemma 5.3. □

7. Connectivity

A matroid $M$ is \textit{weakly round} if there is no pair of sets $A, B$ with union $E(M)$, such that $r_M(A) \leq r(M) - 2$ and $r_M(B) \leq r(M) - 1$. Any matroid of rank at most 2 is clearly weakly round. This is a variation on \textit{roundness}, a notion equivalent to infinite vertical connectivity introduced by Kung [6] under the name of ‘non-splitting’. Weak roundness is preserved by contraction; the following lemma is easily proved, and we use it freely.

\textbf{Lemma 7.1.} If $M$ is a weakly round matroid, and $e \in E(M)$, then $M/e$ is weakly round.

The first step in our proof of the main theorems will be to reduce to the weakly round case; the next two lemmas give this reduction.

\textbf{Lemma 7.2.} If $M$ is a matroid, then $M$ has a weakly round restriction $N$ such that $\varepsilon(N) \geq \varphi^{r(M)} \varepsilon(M)$, where $\varphi = \frac{1}{2}(1 + \sqrt{5})$.

\textbf{Proof.} We may assume that $M$ is not weakly round, so $r(M) > 2$, and there are sets $A, B$ of $M$ such that $r_M(A) = r(M) - 2$, $r_M(B) = r(M) - 1$, and $E(M) = A \cup B$. Now, since $\varphi^{-1} + \varphi^{-2} = 1$, either $\varepsilon(M|A) \geq \varphi^{-2} \varepsilon(M)$ or $\varepsilon(M|B) \geq \varphi^{-1} \varepsilon(M)$; in the first case, by induction $M|A$ has a weakly round restriction $N$ with $\varepsilon(N) \geq \varphi^{r(N)-r(M)} \varepsilon(M|A) \geq \varphi^{r(N)-r(M)+2} \varphi^{-2} \varepsilon(M) = \varphi^{r(M)} \varepsilon(M)$, giving the result. The second case is similar. □

\textbf{Lemma 7.3.} Let $q$ be a prime-power, and $k \geq 0$ be an integer. If $\mathcal{M}$ is a base-$q$ exponentially dense minor-closed class of matroids that contains $(q,k)$-overfull matroids of arbitrarily large rank, then $\mathcal{M}$ contains weakly round, $(q,k)$-overfull matroids of arbitrarily large rank.

\textbf{Proof.} Note that $\varphi < 2 \leq q$; by the Growth Rate Theorem, there is an integer $t > 0$ such that

$$\varepsilon(M) \leq \left( \frac{q}{\varphi} \right)^t \frac{q^{r(M)+k} - 1}{q - 1} - \frac{q^{2k} - 1}{q^2 - 1},$$

for all $M \in \mathcal{M}$.

For any integer $n > 0$, consider a $(q,k)$-overfull matroid $M \in \mathcal{M}$ with rank at least $n + t$. By Lemma 7.2, $M$ has a weakly round restriction
Lemma 8.1. There is an integer-valued function $f_{8.1}(n, q, t, \ell)$ so that, for any prime power $q$, and integers $n \geq 1, \ell \geq 2$ and $t \geq 0$, if $M \in \text{EX}(U_{2,\ell+2})$ is a weakly round matroid with a $\text{PG}(f_{8.1}(n, q, t, \ell) - 1, q)$-minor, and $T$ is a restriction of $M$ of rank at most $t$, then there is a minor $N$ of $M$ of rank at least $n$, such that $T$ is a restriction of $N$, and $N$ has a $\text{PG}(r(N) - 1, q)$-restriction.

Proof. Let $n \geq 1, \ell \geq 2$ and $t \geq 0$ be integers. Let $n' = \max(n, t + 1)$, and set $f_{8.1}(n, q, t, \ell)$ to be an integer $m$ such that $m \geq 2t$, and

$$\frac{q^{m} - 1}{q - 1} \geq \alpha_{2,2}(n', q - \frac{1}{2}, \ell) \left(\ell(q - \frac{1}{2})\right)^{t} (q - \frac{1}{2})^{m}.$$

Let $M \in \text{EX}(U_{2,\ell+2})$ be a weakly round matroid with a $\text{PG}(m - 1, q)$-minor $S = M/\cap D$, where $r(S) = r(M) - r_{M}(C)$. Let $T$ be a restriction of $M$ of rank at most $t$; we show that the required minor exists.

8.1.1. There is a weakly round minor $M_{1}$ of $M$, such that $T$ is a restriction of $M_{1}$, and $M_{1}$ has a $\text{PG}(n' - 1, q)$-restriction $R_{1}$.

Proof of claim: Let $C' \subseteq C$ be maximal such that $T$ is a restriction of $M/C'$, and let $M' = M/C'$. Maximal implies that $C - C' \subseteq \text{cl}_{M'}(E(T))$, so $r_{M'}(C - C') \leq t$. Now, $r_{M'}(E(S)) = r(S) + r_{M'}(C - C') \leq m + t$. Therefore,

$$\varepsilon_{M'}(E(S)) = \frac{q^{m} - 1}{q - 1} \geq \alpha_{2,2}(n', q - \frac{1}{2}, \ell)\ell^{t}(q - \frac{1}{2})^{-t}(q - \frac{1}{2})^{m+t} \geq \alpha_{2,2}(n', q - \frac{1}{2}, \ell)(\ell(q - \frac{1}{2})^{-1})^{t}(q - \frac{1}{2})^{r_{M'}(E(S))}.$$
By Lemma 2.3 applied to $E(S)$ and $E(T)$, with $\mu = q - \frac{1}{2}$, there is a set $A \subseteq E(S)$, skew to $E(T)$ in $M'$, such that

$$\varepsilon(M'|A) \geq \alpha_{2,2}(n', q - \frac{1}{2}, \ell)(q - \frac{1}{2})^{r(M'|A)}.$$ 

Therefore, Lemma 2.2 implies that $M'|A$ has a PG($n' - 1, q'$)-minor $R_1 = (M'|A)/C_1 \backslash D_1$, for some $q' > q - \frac{1}{2}$. Let $M_1 = M'/C_1$. The set $A$ is skew to $E(T)$ in $M'$, and therefore also skew to $C - C'$, so $M'|A = (M'/(C - C'))|A = S|A$, so $M'|A$ is GF($q$)-representable, and so is its minor $R_1$. Thus, $q' = q$, and $R_1$ is a PG($n' - 1, q$)-restriction of $M_1$. Moreover, $C_1 \subseteq A$, so $C_1$ is skew to $E(T)$ in $M'$, and therefore $M_1$ has $T$ as a restriction. The matroid $M_1$ is a contraction-minor of $M$, so is weakly round, and thus satisfies the claim. □

Let $M_2$ be a minor-minimal matroid such that:

- $M_2$ is a weakly round minor of $M_1$, and
- $T$ and $R_1$ are both restrictions of $M_2$.

If $r(R_1) = r(M_2)$, then $N = M_2$ is the required minor of $M$. We may therefore assume that $r(M_2) > r(R_1) = n'$. We have $r(T) \leq t \leq n' - 1 \leq r(M_2) - 2$, so by weak roundness of $M_2$, there is some $e \in E(M_2)$ spanned by neither $E(T)$ nor $E(R_1)$, contradicting minimality of $M_2$. □

9. Critical elements

An element $e$ in a ($q, k$)-overfull matroid $M$ is called ($q, k$)-critical if $M/e$ is not ($q, k$)-overfull.

**Lemma 9.1.** Let $q$ be a prime power and $k \geq 0$ be an integer. If $e$ is a ($q, k$)-critical element in a ($q, k$)-overfull matroid $M$, then either

(i) $e$ is contained in a line with at least $q^2 + 2$ points, or
(ii) $e$ is contained in $\frac{q^{2k-1}}{q-1} + 1$ lines, each with at least $q + 2$ points.

**Proof.** Suppose otherwise. Let $\mathcal{L}$ be the set of all lines of $M$ containing $e$, and let $\mathcal{L}_1$ be the set of the min($|\mathcal{L}|, \frac{q^{2k-1}}{q-1}$) longest lines in $\mathcal{L}$. Every line in $\mathcal{L} - \mathcal{L}_1$ has at most $q + 1$ points and every line in $\mathcal{L}_1$ has at most
$q^2 + 1$ points, so

\[
\varepsilon(M) \leq 1 + q|\mathcal{L}| + (q^2 - q)|\mathcal{L}_1| \\
\leq 1 + q\varepsilon(M/e) + (q^2 - q)q^{2k - 1} \frac{q^2 - 1}{q^2 - 1} \\
\leq 1 + q \left( q^{r(M)+k-1} \frac{q^2 - 1}{q - 1} - q^{2k - 1} \frac{q^2 - 1}{q^2 - 1} \right) + (q^2 - q)q^{2k - 1} \frac{q^2 - 1}{q^2 - 1} \\
= \frac{q^{r(M)+k-1} - 1}{q - 1} + q^{2k - 1} \frac{q^2 - 1}{q^2 - 1},
\]

contradicting the fact that $M$ is $(q, k)$-overfull. \(\square\)

The following result shows that a large number of $(q, k)$-critical elements gives a denser minor.

**Lemma 9.2.** There is an integer-valued function $f_{9.2}(n, q, k)$ so that, for any prime power $q$, and integers $k \geq 0$, $n \geq k+1$, if $m \geq f_{9.2}(n, q, k)$ is an integer, and $M \in \text{EX}(U_{2,q^2+q+1})$ is a $(q, k)$-overfull, weakly round matroid such that

- $M$ has a PG$(m-1, q)$-minor, and
- $M$ has a rank-$m$ set of $(q, k)$-critical elements,

then $M$ has a rank-$n$, $(q, k+1)$-full minor with a $U_{2,q^2+1}$-restriction.

**Proof.** Let $q$ be a prime power, and $k \geq 0$ and $n \geq 2$ be integers. Let $n' = \max(k + 8, n + k + 1)$, let $d = f_{5.2}(q, k)$, let $t = d(d + 1) + k + 6$, let $s = \frac{q^{2k-1}}{q^2 - 1} + 1$, and set $f_{9.2}(n, q, k) = f_{8.1}(n', q, t(s + 1), q^2 + q - 1)$.

Let $m \geq f_{9.2}(n, q, k)$ be an integer, and let $M \in \text{EX}(U_{2,q^2+q+1})$ be a $(q, k)$-overfull, weakly round matroid with a PG$(m-1, q)$-minor and a $t$-element independent set $I$ of $(q, k)$-critical elements (note that $t \leq m$). We will show that $M$ has the required minor.

By Lemma 9.1, for each element $e \in I$, there is a set $\mathcal{L}_e$ of lines containing $e$ such that either $|\mathcal{L}_e| = 1$ and the single line in $\mathcal{L}_e$ has $q^2 + 2$ points, or $|\mathcal{L}_e| = \frac{q^{2k-1}}{q^2 - 1} + 1$ and each line in $\mathcal{L}_e$ has at least $q + 2$ points. There is a restriction $K$ of $M$ with rank at most $t(s + 1)$ that contains all the lines $(\mathcal{L}_e : e \in I)$. By Lemma 8.1, $M$ has a minor $M_1$ of rank at least $n'$ that has a PG$(r(M_1) - 1, q)$-restriction $R_1$, and has $K$ as a restriction. By Lemma 4.1, $M_1$ has at most one line containing $q^2 + 2$ points.

**9.2.1.** There is a $(t - 5)$-element subset $I_1$ of $I$ such that, for each $e \in I_1$, we have $r_K(\bigcup \mathcal{L}_e) \geq k + 2$.

**Proof of claim:** Note that $|I| = t \geq 5$. If $k = 0$, then every $e \in I$ satisfies the required condition, so an arbitrary $(t - 5)$-subset of $I$ will
do; we may thus assume that \( k \geq 1 \). Since \( K \) contains at most one line with at least \( q^2 + 2 \) points, there are at most two elements \( e \in I \) with \( |L_e| = 1 \). If the claim fails, there is therefore an 4-element subset \( I_2 \) of \( I \) such that \( |L_e| = \frac{2^{2k} - 1}{q - 1} + 1 \) and \( r_K(\cup L_e) \leq k + 1 \) for all \( e \in I_2 \).

For each \( e \in I_2 \), let \( F_e = \text{cl}_K(\cup L_e) \). Then \( (K|F_e)/e \) has rank at most \( k \) and has more than \( \frac{2^{2k} - 1}{q - 1} \) points. Since \( k \geq 1 \), this matroid has rank at least 2. Moreover, \( M_1/e \) has rank at least \( n' - 1 \geq k + 7 \) and has a \( \text{PG}\left(r(M_1/e) - 1, q\right) \)-restriction, so, by Lemma 4.2, \( r((K|F_e)/e) = 2 \). Hence, \( k \geq 2 \), \( F_e \) is a rank-3 set containing at least \( q^2 + 2 \) lines through \( e \), each with at least \( q + 2 \) points, and \( (K|F_e)/e \) is a rank-2 set containing at least \( q^2 + 2 \) points.

Let \( a \in I_2 \); since \( r_{M_1}(I_2) = 4 > r_{M_1}(F_a) \), there is some \( b \in I_2 - F_a \). Now, \( M_1/b \) has a line \( L = \text{cl}_{M_1/b}(F_b - \{b\}) \) containing at least \( q^2 + 2 \) points, and \( (M_1/b)|F_a \) is a rank-3 matroid with at least \( 1 + (q+1)(q^2+2) \) points, and therefore at least \( 1 + (q+1)(q^2+2) - (q^2+q) > q^2+q+1 \) points outside \( L \). However, \( M_1/b \) has rank at least \( k + 7 \), and has a \( \text{PG}\left(r(M_1/b) - 1, q\right) \)-restriction containing at most \( q^2 + q + 1 \) points in \( F_a - L \), so we obtain a contradiction to Lemma 4.1.

\[ 9.2.2. \ M_1 \ has \ an \ R_1\text{-unstable \ set \ of \ size} \ k + 1. \]

\textbf{Proof of claim.} Suppose otherwise. By Lemma 5.2, there is a flat \( F \) of \( R_1 \) with rank at most \( k \) such that \( \varepsilon(M_1/F) \leq \varepsilon(R_1/F) + f_{5,2}(q, k) = \varepsilon(R_1/F) + d \). Let \( M_2 = M_1/F \); the matroid \( M_2 \) has a \( \text{PG}\left(r(M_2) - 1, q\right) \)-restriction \( R_2 \), and satisfies \( E(M_2) = E(R_2) \cup D \), where \( |D| \leq d \).

Let \( I_2 \subseteq I_1 \) be a set of size of size \( |I_1| - k \) that is independent in \( M_2 \); note that \( |I_2| \geq d(d+1) + 1 \). For each \( e \in I_2 \), we have \( r_{M_2}(\cup L_e) \geq (k+2) - k = 2 \), so \( e \) is contained in a line \( L_e \) with at least \( q + 2 \) points in \( M_2 \).

Let \( L = \{L_e : e \in I_2\} \). Each \( L_e \) contains \( e \), and at most one other point in \( I_2 \), so \( |L| \geq \frac{1}{2}|I_2| > \binom{d+1}{2} \). Each line in \( L \) contains \( q + 2 \) points, so must contain a point of \( M_2 \setminus E(R_2) \). However, \( |M_2 \setminus E(R_2)| \leq d \), so there are at most \( \binom{d}{2} \) lines of \( M_2 \) containing two points of \( M_2 \setminus E(R_2) \), and by Lemma 3.2, we may assume that there are at most \( d \) lines of \( M_2 \) containing \( q + 2 \) points, but just one point of \( M_2 \setminus E(R_2) \). This gives \( |L| \leq d + \binom{d}{2} = \binom{d+1}{2} \), a contradiction.

Since \( r(M_1) \geq n' \geq n + k + 1 \), we get the required minor \( N \) from the above claim and Lemma 5.3.

\[ 10. \ \text{The main theorems} \]

The following result implies Theorems 1.2 and 1.3:
Theorem 10.1. Let $q$ be a prime power, and let $\mathcal{M} \subseteq \text{EX}(U_{2,q^2+q+1})$ be a base-$q$ exponentially dense minor-closed class of matroids. There is an integer $k \geq 0$ such that

$$h_M(n) = \frac{q^{n+k} - 1}{q - 1} - \frac{q^{2k} - 1}{q^2 - 1}$$

for all sufficiently large $n$. Moreover, if $\mathcal{M} \subseteq \text{EX}(U_{2,q^2+1})$, then $k = 0$.

Proof. By the Growth Rate Theorem, $\mathcal{M}$ contains all projective geometries over $\text{GF}(q)$ and, hence, $\mathcal{M}$ contains $(q,0)$-full matroids of every rank. We may assume that there are $(q,0)$-full matroids of arbitrarily large rank, since otherwise the theorem holds. By the Growth Rate Theorem, there is a maximum integer $k \geq 0$ such that $\mathcal{M}$ contains $(q,k)$-overfull matroids of arbitrarily large rank, and there is an integer $s \geq 0$ such that $\text{PG}(s-1,q') \notin \mathcal{M}$ for all $q' > q$.

To prove the result, it suffices to show that, for all $n > k + 1$, there is a rank-$n$ matroid $M \in \mathcal{M}$ that is $(q,k)$-full and has a $U_{2,q^2+1}$-restriction. Suppose for a contradiction that $n > k + 1$ is an integer for which this $M$ does not exist.

Let $m = f_{g_2}(n,q,k)$, and $m_4 = \max(m + 1, s, f_{6,1}(n,q,k))$. Let $m_3$ be an integer such that

$$q^{m_3} - 1 > \alpha_{2,2}(m_4, q - \frac{1}{2}, q^2 + q - 1) \left( \frac{q^2 + q - 1}{q - \frac{3}{2}} \right)^m (q - \frac{1}{2})^{m_3 + m - 1}. $$

Let $m_2 = \max(s,m_3m)$, and choose an integer $m_1 > s$ such that

$$\alpha_{2,2}(m_2, q - \frac{1}{2}, q^2 + q - 1)(q - \frac{1}{2})^{r} \leq \frac{q^{r+k} - 1}{q - 1} - \frac{q^{2k} - 1}{q^2 - 1}$$

for all $r \geq m_1$. By Lemma 7.3, $\mathcal{M}$ contains weakly round, $(q,k)$-overfull matroids of arbitrarily large rank; let $M_1 \in \mathcal{M}$ be a weakly round, $(q,k)$-overfull matroid with rank at least $m_1$. By Lemma 2.2, $M_1$ has a PG($m_2 - 1,q'$) minor $N_1$ for some $q' > q - \frac{1}{2}$; since $m_2 \geq s$, we have $q' = q$. Let $I_1$ be an independent set of $M_1$ such that $N_1$ is a spanning restriction of $M_1/I_1$, and choose $J_1 \subseteq I_1$ maximal such that $M_1/J_1$ is $(q,k)$-overfull.

Let $M_2 = M_1/J_1$ and let $I_2 = I_1 - J_1$. By our choice of $J_1$, each element in $I_2$ is $(q,k)$-critical in $M_2$. Since $m_2 \geq m$, Lemma 9.2 gives $|I_2| < m$. Choose a collection $(F_1, \ldots, F_m)$ of mutually skew rank-$m_3$ flats in the projective geometry $N_1$; each $F_i$ satisfies $r(M_2|F_i) \leq m_3 + m - 1$ and $\varepsilon(M_2|F_i) = \frac{q^{m_3} - 1}{q - 1}$. By our choice of $m_3$, and by Lemma 2.3 with $\mu = q - \frac{1}{2}$ for each $i \in \{1, \ldots, m\}$, there is a flat $F'_i \subseteq F_i$ of $M_2$ that is skew to $I_2$ in $M_2$, and satisfies $\varepsilon(M_2|F'_i) \geq \alpha_{2,2}(m_4, q - \frac{1}{2}, q^2 + q - 1)(q - \frac{1}{2})^{r_{m_2}}(F'_i)$. Note that, since the sets $(F'_1, \ldots, F'_m)$ are
mutually skew in $M_2/I_2$ and each of these sets is skew to $I_2$ in $M_2$, the flats $(F'_1, \ldots, F'_m)$ are mutually skew in $M_2$.

By Lemma 2.2, $M_2|F'_i$ has a $\text{PG}(m_4 - 1, q')$ minor $P_i$ for some $q' > q - \frac{1}{2}$; since $m_4 \geq s$, we have $q' = q$. Let $X_i$ be an independent set of $M_2|F'_i$ such that $P_i$ is a spanning restriction of $M_2/X_i$. Now choose $Z \subseteq X_1 \cup \cdots \cup X_m$ maximal such that $M_2/Z$ is $(q, k)$-overfull. Let $M_3 = M_2/Z$. Each element of $X_1 \cup \cdots \cup X_s - Z$ is $(q, k)$-critical in $M_3$, and $P_i$ is a minor of $M_3$ for each $i$. The $X_i$ are mutually skew in $M_3$ and hence pairwise disjoint; thus, by Lemma 9.2, there exists $i_0 \in \{1, \ldots, m\}$ such that $X_{i_0} - Z = \emptyset$ and, hence, $P_{i_0}$ is a restriction of $M_3$; let $R = P_{i_0}$.

Choose a minor $M_4$ of $M_3$ that is minimal such that:

- $M_4$ is weakly round, and $(q, k)$-overfull,
- $M_4$ has $R$ as a restriction.

By Lemma 6.1, $r(M_4) > r(R)$. Every element of $E(M_4) - \text{cl}_{M_4}(E(R))$ is $(q, k)$-critical and, since $M_4$ is weakly round, $r(M_4 \setminus \text{cl}_{M_4}(E(R))) \geq r(M_4) - 2 \geq m_4 - 1 \geq m$. We now get a contradiction from Lemma 9.2.

\[\square\]

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**References**


Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada