R-TRANSFORMS IN FREE PROBABILITY

ALEXANDRU NICA

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14. The S-transform

Recall that if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and if $a \in \mathcal{A}$ is such that $\varphi(a) \neq 0$, then the S-transform of a is defined to be the series:

$$S_a(z) \; := \; \frac{1}{z} R_a^{<-1>}(z) \; = \; \frac{1+z}{z} M_a^{<-1>}(z) \tag{1}$$

(cf. Section 13, Definition 13.9). The S-transform was introduced (by Voiculescu, in 1987), for its property of transforming the multiplication of free elements into a multiplication of power series. More precisely: if $a, b \in \mathcal{A}$ are free, then it will turn out that the S-transform S_{ab} is the product of the series S_a and S_b.

In the approach taken by these notes, the natural path is to derive the multiplicativity of the S-transform by studying the group of invertible elements with respect to the operation $\boxed{\star}$.

Notation 14.1. Let s be a positive integer, and consider the operation $\boxed{\star}$ on Θ_s (as in Section 12). We will denote by $\Theta_s^{(inv)}$ the set of series $f \in \Theta_s$ which are invertible with respect to $\boxed{\star}$.

From the properties of $\boxed{\star}$ discussed in Section 12, it is clear that $(\Theta_s^{(inv)}, \boxed{\star})$ is a group. It is very easy to determine which series $f \in \Theta_s$ belong to $\Theta_s^{(inv)}$: as shown in Proposition 12.8, the necessary and sufficient condition for that to happen is that all the linear coefficients of f are different from 0. But this does not mean at all that we know a nice description of what $(\Theta_s^{(inv)}, \boxed{\star})$ is, as group structure. The main theorem of the present section is exactly concerned with the clarification of the structure of this group, in the case $s = 1$. The problem of finding an analogous theorem for $s \geq 2$ is not solved (and it is in fact not so clear if one should expect to have a nice solution).

In order to state the theorem, we will also introduce the following notations.

Notations 14.2. 1) We denote by Γ_1 the set of all power series $u(z) = \sum_{n=0}^{\infty} u_n z^n$ with the property that the constant term u_0 of u is not equal to 0. On Γ_1 we consider the group structure $(\Gamma_1, \cdot)$ given by the usual multiplication of power series.

2) For $f \in \Theta_1^{(inv)}$ we denote:

$$[\mathcal{F}(f)\ ](z) \ = \ \frac{1}{z} f^{<-1>}(z), \tag{2}$$

where $f^{<-1>}$ is the inverse of f under composition. In this way we obtain a map $\mathcal{F} : \Theta_1^{(inv)} \to \Gamma_1$ (indeed, denoting by α_1 the linear coefficient of f, it is clear that the constant coefficient of $\mathcal{F}(f)$ is $\alpha_1^{-1} \neq 0$, hence that $\mathcal{F}(f) \in \Gamma_1$).

Theorem 14.3. *The function $\mathcal{F}$ introduced in the Notation 14.2.2 is a group isomorphism between $(\ \Theta_1^{(inv)}, \boxed{\star}\ )$ and $(\Gamma_1, \cdot)$.*

Before starting to discuss the proof of this theorem, let us notice that it has as immediate corollary the multiplicativity of the S-transform.

Corollary 14.4. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a, b be in $\mathcal{A}$. If a is free from b, then:*

$$S_{ab}(z) \ = \ S_a(z) \cdot S_b(z). \tag{3}$$

Proof. As is clear by comparing the Equations (1) and (2), we have the relation:

$$\mathcal{F}(\ R_x\ ) \ = \ S_x, \tag{4}$$

holding for every $x \in \mathcal{A}$ such that $\varphi(x) \neq 0$. Hence we can write:

$$\begin{aligned} S_{ab} \ &= \ \mathcal{F}(\ R_{ab}\ ) \\ &= \ \mathcal{F}(\ R_a \boxed{\star} R_b\ ) \quad \text{(by Proposition 12.3)} \\ &= \ \mathcal{F}(\ R_a\ ) \cdot \mathcal{F}(\ R_b\ ) \quad \text{(by Theorem 14.3)} \\ &= \ S_a \cdot S_b. \end{aligned}$$

□

On our way towards the proof of the Theorem 14.3, a key point will be to obtain a formula which is similar in nature to the functional equation of the R-transform (in the case of 1 variable), but where we consider boxed convolution with a series different from Zeta. To be more precise: the case $s = 1$ of Theorem 13.2 says that for two series $f, g \in \Theta_1$, the relation $g = f \boxed{\star}$ Zeta is equivalent to $g = f \circ (\ z(1+g)\ )$.

If f is invertible under composition, we thus obtain that $f^{<-1>} \circ g = z(1+g)$, or writing only in terms of f:

$$f^{<-1>} \circ (f \boxed{\star} \text{Zeta}) \; = \; z(\, 1 + f \boxed{\star}\, Zeta\,). \tag{5}$$

In the following Sections 14.5 - 14.8, we will discuss a generalization of (5) to the case when Zeta is replaced by an arbitrary series $h \in \Theta_1$.

Notation 14.5. Let $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ be a series in Θ_1. For every $n \geq 1$ and $\pi = \{B_1, \ldots, B_r\} \in NC(n)$, we denote

$$\text{coef}_\pi(f) \; := \; \alpha_{|B_1|} \cdots \alpha_{|B_r|}. \tag{6}$$

This is a modification of the coefficient notation set in Section 11.4, which is more suitable for series of one variable.

Definition 14.6. Let f and h be series in Θ_1. The *incomplete boxed convolution* of f and h, denoted by $f \overset{\vee}{\boxed{\star}} h$, is the series

$$(f \overset{\vee}{\boxed{\star}} h)(z) \; := \; \sum_{n=1}^{\infty} \lambda_n z^n \in \Theta_1,$$

where for every $n \geq 1$ we set:

$$\lambda_n \; = \sum_{\substack{\pi \in NC(n) \text{ such} \\ \text{that } (1) \text{ is} \\ \text{a block of } \pi}} \text{coef}_\pi(f) \cdot \text{coef}_{K(\pi)}(h). \tag{7}$$

Proposition 14.7. *Let f and h be series in Θ_1. We denote by α_1 the coefficient of z in f. Suppose that f is invertible under composition, i.e. that $\alpha_1 \neq 0$. Then we have that:*

$$f^{<-1>} \circ (f \boxed{\star}\, h) \; = \; \frac{1}{\alpha_1}(f \overset{\vee}{\boxed{\star}} h). \tag{8}$$

Proof. We will show, equivalently, that

$$f \boxed{\star}\, h \; = \; f \circ (\frac{1}{\alpha_1}(f \overset{\vee}{\boxed{\star}} h)\,). \tag{9}$$

We fix a positive integer m, and we will verify the equality of the coefficients of order m in the series appearing on the two sides of Equation(9). The verification is similar to the part (b) in the proof of Theorem 13.2, for this reason we will not insist to give all the details.

Let us write explicitly:

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n, \;\; (f \overset{\vee}{\boxed{\star}} h)(z) \; = \; \sum_{n=1}^{\infty} \lambda_n z^n.$$

The coefficient of order m in the series on the right-hand side of (9) is expressed in terms of the α_n's and the λ_n's as:

$$\sum_{n=1}^{m} \sum_{\substack{i_1,\ldots,i_n\geq 1 \\ i_1+\cdots+i_n=m}} \alpha_n \alpha_1^{-n} \cdot \lambda_{i_1}\cdots\lambda_{i_n}. \tag{10}$$

Let us now look at the coefficient of order m on the left-hand side of (9). By the definition of $\boxed{\star}$, this is equal to:

$$\sum_{\pi\in NC(m)} \mathrm{coef}_{\pi}(f)\cdot \mathrm{coef}_{K(\pi)}(h).$$

The summation over $NC(m)$ can be detailed, by classifying the partitions in $NC(m)$ according to their first block. When this is done, we obtain the following expression (analogous to Equation (9) of Section 13):

$$\sum_{n=1}^{m} \sum_{1=b_1<\cdots<b_n\leq m} \sum_{\substack{\pi\in NC(m)\ s.t. \\ \{b_1,\ldots,b_n\} \\ is\ block\ of\ \pi}} \mathrm{coef}_{\pi}(f)\cdot \mathrm{coef}_{K(\pi)}(h). \tag{11}$$

But if a partition $\pi\in NC(m)$ is subjected to the condition of having a prescribed block $B=\{b_1,\ldots,b_n\}$, with $1=b_1<\cdots<b_n\leq m$, then knowing π is equivalent to knowing its restrictions to the spaces left between the consecutive elements of B. This time we proceed slightly differently from what we did in the proof of Theorem 13.2, and we denote: $\pi_1=$ the restriction of π to $\{b_1,\ldots,b_2-1\}$, $\pi_2=$ the restriction of π to $\{b_2,\ldots,b_3-1\},\ldots,\pi_n=$ the restriction of π to $\{b_n,\ldots,m\}$. The difference consists in the fact that each of $\pi_1,\ldots,\pi_n$ also has (in addition to containing a union of blocks of π) a block of one element at the left end. The advantage of setting the notations this way is that we get a nice relation when we look at Kreweras complements: as is immediately checked, we have that $K(\pi)$ is just the juxtaposition of the Kreweras complements $K(\pi_1),\ldots,K(\pi_n)$. The relations between π and $\pi_1,\ldots,\pi_n$ lead us to the equations:

$$\mathrm{coef}_{\pi}(f) \ = \ \alpha_n\alpha_1^{-n}\cdot \mathrm{coef}_{\pi_1}(f)\cdots \mathrm{coef}_{\pi_n}(f), \tag{12}$$

and

$$\mathrm{coef}_{K(\pi)}(h) \ = \ \mathrm{coef}_{K(\pi_1)}(h)\cdots \mathrm{coef}_{K(\pi_n)}(h) \tag{13}$$

((12) is the analogue of Equation (10) in Section 13, (13) is an additional formula which we obtain from the relation between Kreweras complements).

We substitute (12) and (13) into (11); we get that the latter quantity is thus equal to:

$$\sum_{n=1}^{m} \sum_{1=b_1<\cdots<b_n\leq m} \alpha_n \alpha_1^{-n} \cdot \sum_{\pi_1,\cdots,\pi_n} \mathrm{coef}_{\pi_1}(f)\cdots \tag{14}$$

$$\cdots \mathrm{coef}_{\pi_n}(f) \cdot \mathrm{coef}_{K(\pi_1)}(h) \cdots \mathrm{coef}_{K(\pi_n)}(h),$$

where $\pi_1 \in NC(b_2 - b_1), \pi_2 \in NC(b_3 - b_2), \ldots, \pi_n \in NC(m - b_n + 1)$ are only subjected to the condition that they start with a one-element block on the left. In (14) we can clearly factor out separate summations over $\pi_1, \ldots, \pi_n$. Due to the way how $\overset{\vee}{\boxed{\star}}$ was defined, we find on the other hand that $\sum_{\pi_1} \mathrm{coef}_{\pi_1}(f)\mathrm{coef}_{K(\pi_1)}(h) = \lambda_{b_2-b_1}, \ldots,$ $\sum_{\pi_n} \mathrm{coef}_{\pi_n}(f)\mathrm{coef}_{K(\pi_n)}(h) = \lambda_{m-b_n+1}$. Hence (14) becomes:

$$\sum_{n=1}^{m} \sum_{1=b_1<\cdots<b_n\leq m} \alpha_n \alpha_1^{-n} \cdot \lambda_{b_2-b_1}\lambda_{b_3-b_2}\cdots\lambda_{m-b_n+1}, \tag{15}$$

and it is an immediate exercise to check the equality between (15) and (10). ☐

Exercise 14.8. Verify that if in the framework of Proposition 14.7 we set $h = \mathrm{Zeta}$, then the Equation (8) reduces to the reformulation (5) of the functional equation for the R-transform.

We can now present the proof of the Theorem 14.3.

Proof. The fact that $\mathcal{F}$ is bijective is immediate, the problem is to prove that the relation

$$\mathcal{F}(f \boxed{\star}\, g) \;=\; \mathcal{F}(f) \cdot \mathcal{F}(g) \tag{16}$$

holds for every $f, g \in \Theta_1^{(inv)}$. By substituting in (16) the explicit definition of $\mathcal{F}$ (given in Equation (2) above) we see that what we have to show is:

$$z \cdot (f \boxed{\star}\, g)^{<-1>}(z) \;=\; f^{<-1>}(z) \cdot g^{<-1>}(z), \;\; \forall\, f, g \in \Theta_1^{(inv)}. \tag{17}$$

For the rest of the proof we fix two series in $\Theta_1^{(inv)}$,

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} \beta_n z^n,$$

about which we will show that (17) holds.

In order to eliminate the inverses under composition which appear in (17), we will compose both sides of this equation, on the right, with $f \boxed{\star} g$. The new equation obtained in this way will be equivalent to (17), since we can always go back by composing with $(f \boxed{\star} g)^{<-1>}$.

When composed with $f \boxed{\star} g$ on the right, the left-hand side of (17) becomes:

$$\begin{aligned} &\Big(z \cdot (f \boxed{\star} g)^{<-1>} \Big) \circ (f \boxed{\star} g) \\ = &\Big(z \circ (f \boxed{\star} g)\Big) \cdot \Big((f \boxed{\star} g)^{<-1>} \circ (f \boxed{\star} g) \Big) \\ = &(f \boxed{\star} g)(z) \cdot z. \end{aligned}$$

A similar calculation done on the right-hand side of (17) leads us to the series

$$\Big(f^{<-1>} \circ (f \boxed{\star} g) \Big) \cdot \Big(g^{<-1>} \circ (f \boxed{\star} g) \Big),$$

which, by Proposition 14.7, is equal to $\alpha_1^{-1}\beta_1^{-1} \cdot (f \overset{\vee}{\boxed{\star}} g)(g \overset{\vee}{\boxed{\star}} f)$. The equation equivalent to (17) which we obtain is thus:

$$(f \overset{\vee}{\boxed{\star}} g)(z) \cdot (g \overset{\vee}{\boxed{\star}} f)(z) \;=\; \alpha_1\beta_1 z (f \boxed{\star} g)(z). \tag{18}$$

In order to conclude the proof, we fix a positive integer m, about which we show that the coefficients of z^{m+1} on the two sides of (18) are equal. As is immediately seen, the coefficient of z^{m+1} on the left-hand side of (18) is equal to:

$$\sum_{n=1}^{m} \sum_{\substack{\pi \in NC(n), \\ (1) \textit{ block of } \pi}} \sum_{\substack{\rho \in NC(m+1-n), \\ (1) \textit{ block of } \rho}} \mathrm{coef}_{\pi}(f)\mathrm{coef}_{K(\pi)}(g) \times \tag{19}$$

$$\times \mathrm{coef}_{\rho}(g)\mathrm{coef}_{K(\rho)}(f),$$

while the corresponding coefficient on the right-hand side of (18) is

$$\sum_{\sigma \in NC(m)} \alpha_1\beta_1 \mathrm{coef}_{\sigma}(f)\mathrm{coef}_{K(\sigma)}(g). \tag{20}$$

Now, the point is that there exists a natural bijection between the index sets of the sums in (19) and (20),

$$\bigcup_{\substack{1 \leq n \leq m \\ (disjoint)}} \{\pi \in NC(n) \mid (1) \text{ is block of } \pi\} \times \tag{21}$$

$$\times \{\rho \in NC(m-n+1) \mid (1) \text{ is block of } \rho\} \;\longleftrightarrow\; NC(m)$$

such that whenever $(\pi, \rho) \leftrightarrow \sigma$ by this bijection, the term indexed by (π, ρ) in the sum (19) equals the term indexed by σ in the sum (20) – and even more precisely:

$$\left\{ \begin{array}{rcl} \mathrm{coef}_{\pi}(f)\mathrm{coef}_{K(\rho)}(f) & = & \alpha_1 \mathrm{coef}_{\sigma}(f), \\ \\ \mathrm{coef}_{K(\pi)}(g)\mathrm{coef}_{\rho}(g) & = & \beta_1 \mathrm{coef}_{K(\sigma)}(g). \end{array} \right. \tag{22}$$

From left to right, the description of the bijection (21) goes as follows: start with $1 \leq n \leq m$, $\pi \in NC(n)$ such that (1) is a block of π, $\rho \in NC(m+1-n)$ such that (1) is a block of ρ. Denote by $\pi_o \in NC(n-1)$ the partition obtained by deleting the one-element block (1) of π, and consider on the other hand the Kreweras complement $K(\rho) \in NC(m+1-n)$. Then $\sigma \in NC(m)$ which corresponds by (21) to (π, ρ) is obtained by simply juxtaposing π_o and $K(\rho)$, in this order.

[Numerical example: if $n = 6$, $j = 3$, $\pi = \{(1), (2, 3)\}$, $\rho = \{(1), (2, 4), (3)\}$, then $\sigma = \{(1, 2), (3, 6), (4, 5)\}$.]

If on the other hand one wants to describe the bijection (21) from right to left, this is done as follows: start with $\sigma \in NC(m)$, and denote by n the smallest element of the block of σ containing m. Then each of $\{1, \ldots, n-1\}$ and $\{n, \ldots, m\}$ is a union of blocks of σ, thus σ is obtained as the juxtaposition of two non-crossing partitions $\sigma_1 \in NC(n-1)$ and $\sigma_2 \in NC(m+1-n)$. We let $\pi \in NC(n)$ be the partition obtained by adding a one-element block to the left of σ_1, and we put $\rho = K^{-1}(\sigma_2) \in NC(m+1-n)$ ($K^{-1}(\sigma_2)$ has (1) as a block – this is implied by the fact that 1 and $m+1-n$ are in the same block of σ_2). The pair (π, ρ) obtained in this way is what corresponds to σ by the map (21).

We leave it as an exercise to check that the bijection described in the preceding paragraph also has the following property: if $(\pi, \rho) \leftrightarrow \sigma$ by the bijection, then $K^{-1}(\sigma)$ is the juxtaposition of $K(\pi)$ and ρ_o, where ρ_o denotes the partition obtained by deleting the left-most block (of one element) of the partition ρ.

Finally, let us observe that if $(\pi, \rho) \leftrightarrow \sigma$ by the bijection (21), then the Equations (22) are indeed satisfied. The first of these equations follows directly from how σ is obtained as a juxtaposition of π_o and $K(\rho)$, while the second Equation (22) follows from the analogous property of $K^{-1}(\sigma)$:

$$\mathrm{coef}_{K(\sigma)}(g) \;=\; \mathrm{coef}_{K^{-1}(\sigma)}(g)$$

(because $K(\sigma)$ and $K^{-1}(\sigma)$ are obtained from each other by a cyclic permutation – same argument as in Equation (14) of Section 12)

$$= \operatorname{coef}_{K(\pi)}(g) \cdot \operatorname{coef}_{\rho_o}(g)$$

(because $K^{-1}(\sigma)$ is the juxtaposition of $K(\pi)$ and ρ_o)

$$= \frac{1}{\beta_1} \operatorname{coef}_{K(\pi)}(g) \cdot \operatorname{coef}_{\rho}(g)$$

(due to the relation between ρ and ρ_o). $\square$