R-TRANSFORMS IN FREE PROBABILITY

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11. The R-transform

The lectures in this series are coordinated with the ones given in the parallel course of Roland Speicher. We are thus still dealing with combinatorial aspects of free probability. There is however a difference of point of view:

free cumulants	$\leftrightarrow$	the R-transform
("coefficients")		("power series")

Notations 11.1. 1) The framework we use will be the one of a non-commutative probability space $(\mathcal{A}, \varphi)$, i.e $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional normalized such that $\varphi(I) = 1$. If needed, we will use the Banach algebra framework (where it is assumed that $\mathcal{A}$ is a Banach algebra and that $||\varphi|| = 1$), or the $*$-algebra framework (where it is assumed that $\mathcal{A}$ is a $*$-algebra, and that φ is selfadjoint, in the sense that $\varphi(a^*) = \overline{\varphi(a)}$, $a \in \mathcal{A}$).

2) Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a be an element of $\mathcal{A}$. Then the numbers in the sequence $(\varphi(a^n))_{n=1}^{\infty}$ are called the *moments* of a. In what follows we also want to "hang all these numbers together" in a formal power series which will be denoted by M_a, and is called the *moment series* of a:

$$M_a(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n. \tag{1}$$

3) Suppose now that we are given several elements $a_1, \dots, a_s$ of $\mathcal{A}$. Then the numbers in the family:

$$\Big(\varphi(a_{i_1} \cdots a_{i_n}) \Big)_{n \geq 1, 1 \leq i_1, \dots, i_n \leq s}$$

are called the *joint moments* of $a_1, \dots, a_s$. By analogy with (1), all these numbers are put together in a formal power series, which will be denoted by $M_{a_1,\dots,a_s}$, and is called the *moment series* of $a_1, \dots, a_s$. This is a series in s non-commuting indeterminates $z_1, \dots, z_s$:

$$M_{a_1,\dots,a_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{s} \varphi(a_{i_1} \cdots a_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{2}$$

The series of the form appearing in Equation (2) will play an important role in these lectures, and it is in fact worth making the following:

Notation 11.2. Let s be a positive integer. We denote

$$\Theta_s = \left\{ f \;\middle|\; \begin{array}{l} f(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{s} \alpha_{i_1,\dots i_n} z_{i_1} \cdots z_{i_n} \\ \\ \text{where } \alpha_{i_1,\dots,i_n} \in \mathbb{C} \ (n \geq 1, \ 1 \leq i_1, \dots, i_n \leq s) \end{array} \right\} \tag{3}$$

The R-transform of an s-tuple $a_1, \dots, a_s$ of elements in $\mathcal{A}$ will be defined to be a series in the set Θ_s of (3), having free cumulants as coefficients. Following the notes of Speicher (see his Section 4, Definition 4.5), we will denote by $(k_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ the sequence of multilinear free-cumulant functionals of the space $(\mathcal{A}, \varphi)$.

Definition 11.3. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, let s be a positive integer, and let $a_1, \dots, a_s$ be in $\mathcal{A}$. The *R-transform* of $a_1, \dots, a_s$ is the series $R_{a_1,\dots,a_s} \in \Theta_s$ given by the formula;

$$R_{a_1,\dots,a_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{s} k_n(a_{i_1}, \dots, a_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{4}$$

We will need some specific notation for the operation of "extracting a coefficient" of a series in Θ_s.

Notations 11.4. Let s be a positive integer, and consider a series

$$f(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{s} \alpha_{i_1,\dots i_n} z_{i_1} \cdots z_{i_n}$$

in the set Θ_s of (3).

1) For every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq s$ we denote

$$\alpha_{i_1,\dots,i_n} =: coef_{(i_1,\dots,i_n)}(f) \tag{5}$$

("the coefficient of order $(i_1, \dots, i_n)$" of f).

2) For every $n \geq 1$, every $1 \leq i_1, \dots, i_n \leq s$, and every non-crossing partition $\pi = \{B_1, \dots, B_r\} \in NC(n)$ we denote

$$coef_{(i_1,\dots,i_n);\pi}(f) := \alpha_{(i_1,\dots,i_n)|B_1} \cdots \alpha_{(i_1,\dots,i_n)|B_r}. \tag{6}$$

Remark 11.5. The quantity $coef_{(i_1,\dots,i_n);\pi}(f)$ introduced in Equation (6) isn't hence a true coefficient of f, but a product of such coefficients. The notation in (6) matches the notations of the type "$k_\pi[a_1,\dots,a_n]$" used in the notes of Speicher (see Equation (3) in Section 4 there). Indeed, it is clear that given a non-commutative probability space $(\mathcal{A},\varphi)$ and the elements $a_1,\dots,a_s \in \mathcal{A}$, we have:

$$coef_{(i_1,\dots,i_n)}(R_{a_1,\dots,a_s}) \;=\; k_n(a_{i_1},\dots,a_{i_n}), \tag{7}$$

and more generally:

$$coef_{(i_1,\dots,i_n);\pi}(R_{a_1,\dots,a_s}) \;=\; k_\pi[a_{i_1},\dots,a_{i_n}], \tag{8}$$

for every $n \geq 1$, every $1 \leq i_1,\dots,i_n \leq s$, and every $\pi \in NC(n)$.

Remark 11.6. As explained in the notes of Speicher, the free cumulants represent the analogue in free probability of the concept of cumulants from classical probability. The cumulants of a family of real-valued random variables $X_1,\dots,X_s$ (in classical, commutative sense) are defined as the coefficients of the power series $\log \mathcal{F}(\mu_{X_1,\dots,X_s})$, where $\mu_{X_1,\dots,X_s}$ denotes the joint distribution of $X_1,\dots,X_s$ (a probability measure on $\mathbb{R}^s$), and $\mathcal{F}$ denotes the Fourier transform, or characteristic function, of that distribution.

But on the other hand, the free cumulants are the coefficients of the R-transform (of the s-tuple of non-commutative random variables which we want to consider). Hence the analogy at the level of cumulants leads to the following important statement:

The R-transform is the analogue in free probability of the log of the Fourier transform.

This statement is illustrated for instance by the following theorem ("a freeness criterion in terms of R-transforms"), which can be considered as the fundamental result of the theory of the R-transform.

Theorem 11.7. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and let $a_1,\dots,a_r,b_1,\dots,b_s$ be elements of $\mathcal{A}$. Then the family $\{a_1,\dots,a_r\}$ is free from the family $\{b_1,\dots,b_s\}$ if and only if the following equation holds:*

$$R_{a_1,\dots,a_r,b_1,\dots,b_s}(z_1,\dots,z_r,w_1,\dots,w_s) \tag{9}$$

$$= \; R_{a_1,\dots,a_r}(z_1,\dots,z_r) \;+\; R_{b_1,\dots,b_s}(w_1,\dots,w_s).$$

Proof. A slight modification of the proof of Theorem 6.4 in the notes of Speicher gives that the freeness of $\{a_1,\dots,a_r\}$ and $\{b_1,\dots,b_s\}$ is equivalent to the following fact: $k_n(x_1,\dots,x_n) = 0$ whenever $n \geq 2$,

$x_1, \ldots, x_n \in \{a_1, \ldots, a_r\} \cup \{b_1, \ldots, b_s\}$, and there exist $1 \leq i, j \leq n$ such that $x_i \in \{a_1, \ldots, a_r\}$ and $x_j \in \{b_1, \ldots, b_s\}$. (In words: "freeness is equivalent to the vanishing of all the mixed cumulants".) But in view of the definition of the R-transform, the latter condition on free cumulants translates exactly into the Equation (9). $\square$

Remark 11.8. The Equation (9) in the preceding theorem is the free analogue of the following basic property of the Fourier transform: if $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ are random variables on the same probability space (in classical, commutative sense), and if the family $X_1, \ldots, X_r$ is independent from the family $Y_1, \ldots, Y_s$, then

$$\begin{aligned} (10) \qquad & \mathcal{F}_{X_1,\ldots,X_r,Y_1,\ldots,Y_s}(z_1, \ldots, z_r, w_1, \ldots, w_s) \\ & = \mathcal{F}_{X_1,\ldots,X_r}(z_1, \ldots, z_r) \cdot \mathcal{F}_{Y_1,\ldots,Y_s}(w_1, \ldots, w_s) \end{aligned}$$

(where in (10) we have that $z_1, \ldots, z_r, w_1, \ldots, w_s$ are now commuting complex variables). Indeed, the R-transform is the free analogue of $\log \mathcal{F}$, and applying a log in Equation (10) (in the sense of formal power series, for instance) will bring it to a form which looks very similar to Equation (9) of Theorem 11.7.