Hopf algebra approach to ⊠ and S-transform

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A. The 1-variable framework

The main goal of this part of the talk is to advertise a connection between:

(1) multiplication of free random variables
(operation ⊠, the S-transform – in algebraic framework)
and
(2) symmetric functions.

A.1. Review of multiplication of free random variables

We will use an algebraic framework. By *noncommutative probability space* we simply understand a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ (no $*$-operation required!) and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$.

We also make the notation

$$\mathcal{D}_{\text{alg}} := \{\mu : \mathbb{C}[X] \to \mathbb{C} \mid \mu \text{ linear}, \, \mu(1) = 1\}.$$

We think of the functionals in $\mathcal{D}_{\text{alg}}$ as of some (algebraic caricatures of) distributions. If $(\mathcal{A}, \varphi)$ is a noncommutative probability space and if $a \in \mathcal{A}$, then the *distribution* of a (with respect to φ) is the functional $\mu \in \mathcal{D}_{\text{alg}}$ uniquely determined by the requirement that

$$\mu(X^n) = \varphi(a^n), \quad \forall \, n \in \mathbb{N}.$$

The main role in this part of the talk is played by a binary operation ⊠ on $\mathcal{D}_{\text{alg}}$, called *free multiplicative convolution*, which follows the multiplication of free elements in a noncommutative probability space. The precise statement for what this means is that, for every $\mu, \nu \in \mathcal{D}_{\text{alg}}$, the following happens:

$$\left\{ \begin{array}{l} \text{if } (\mathcal{A}, \varphi) \text{ is a noncommutative probability space} \\ \quad \text{and if } a, b \in \mathcal{A} \text{ are such that } a \text{ is free from } b \text{ in } (\mathcal{A}, \varphi) \\ \quad \text{and if } a \text{ has distribution } \mu \text{ while } b \text{ has distribution } \nu \\ \text{then the product } ab \in \mathcal{A} \text{ has distribution } \mu \boxtimes \nu. \end{array} \right.$$

The above rule and the rules of free independence allow us to write concrete formulas for the moments of $\mu \boxtimes \nu$, expressing them as polynomials in the moments of μ and of ν. In the simplest case of the moment of order 1, we just have

$$(\mu \boxtimes \nu)(X) = \mu(X) \cdot \nu(X).$$

Indeed, with a, b as above (free elements in $(\mathcal{A}, \varphi)$ with distributions μ and ν) we write

$$(\mu \boxtimes \nu)(X) = \varphi(ab) = \varphi(a) \cdot \varphi(b) = \mu(X) \cdot \nu(X).$$

Here is the analogous calculation for the moment of order 2:

$$\begin{aligned}(\mu \boxtimes \nu)(X^2) &= \varphi(abab) \\ &= \big(\varphi(a)\big)^2 \varphi(b^2) + \varphi(a^2)\big(\varphi(b)\big)^2 - \big(\varphi(a)\big)^2\big(\varphi(b)\big)^2 \\ &= \big(\mu(X)\big)^2 \nu(X^2) + \mu(X^2)\big(\nu(X)\big)^2 - \big(\mu(X)\big)^2\big(\nu(X)\big)^2.\end{aligned}$$

(The formulas get, of course, more and more stuffy as we move towards higher moments.)

Now another notation: we put

$$\mathcal{G} := \{\mu \in \mathcal{D}_{\text{alg}} \mid \mu(X) = 1\}.$$

From the formula for $(\mu \boxtimes \nu)(X)$ it follows that $(\mathcal{G}, \boxtimes)$ is a semigroup. It can be proved without much difficulty that $(\mathcal{G}, \boxtimes)$ is in fact a commutative group. A natural question around the multiplication of free elements is then the following.

Natural Question: Identify what is the group $(\mathcal{G}, \boxtimes)$.

This question was solved by Voiculescu [5] by using the S-transform. In order to describe this, let us denote by $\mathcal{F}$ the set of formal power series of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} \gamma_n z^n, \quad \text{with } \gamma_n\text{'s in } \mathbb{C}.$$

Every $\mu \in \mathcal{G}$ has an S-stransform $S_\mu \in \mathcal{F}$, and it turns out that

$$S_{\mu \boxtimes \nu} = S_\mu \cdot S_\nu, \quad \forall\, \mu, \nu \in \mathcal{G}.$$

Once the above multiplicativity is established, it follows immediately that the map $\mu \mapsto S_\mu$ gives a group isomorphism $(\mathcal{G}, \boxtimes) \simeq (\mathcal{F}, \cdot)$, where the operation considered on $\mathcal{F}$ is plain multiplication of series.

The concrete formula for S_μ is not complicated to state:

$$S_\mu(z) = \frac{1+z}{z} M_\mu^{<-1>}(z),$$

where M_μ is the moment series of μ (that is, $M_\mu(z) = \sum_{n=1}^{\infty} \mu(X^n) z^n$) and the inverse of M_μ is taken with respect to the operation of composing power series.

A final remark concerning the operation $\boxtimes$ on $\mathcal{D}_{\text{alg}}$: this operation can also be addressed by using another important transform of free probability, the R-*transform*. For $\mu \in \mathcal{G}$, the R-transform R_μ is a series of the form

$$R_\mu(z) = \sum_{n=1}^{\infty} \alpha_n z^n,$$

where the α_n's are called "free cumulants" of μ. (The condition that $\mu \in \mathcal{G}$ forces $\alpha_1 = \mu(X) = 1$.) One has nice combinatorial formulas, using summations over non-crossing partitions, which describe the coefficients of $R_{\mu \boxtimes \nu}$ in terms of the coefficients of R_μ and of R_ν. Using this approach is less efficient than the S-transform, but has the advantage that formulas for the R-transform have straightforward extensions to *joint distributions of k-tuples* of elements in a noncommutative probability space (as found in [3], see also Lecture 17 in [4]). This will be very useful for part B of the talk.

A.2. Review of symmetric functions

We use the (rather customary) notation Sym for the algebra of symmetric polynomials in a countable family of commuting indeterminates $(t_i)_{i=1}^{\infty}$. The fundamental theorem of symmetric functions says that Sym can be identified as

$$\mathrm{Sym} \simeq \mathbb{C}[e_1, e_2, \ldots, e_n, \ldots]$$

where the e_n's are the *elementary symmetric functions:*

$$e_1 = \sum_{i=1}^{\infty} t_i, \quad e_2 = \sum_{1 \leq i < j}^{\infty} t_i t_j, \ldots \ldots, e_n := \sum_{1 \leq i_1 < \cdots < i_n} t_{i_1} t_{i_2} \cdots t_{i_n}, \ldots$$

We will also use the notation

$$\mathbb{X}(\mathrm{Sym}) := \{\chi : \mathrm{Sym} \to \mathbb{C} \mid \chi \text{ is a character}\}$$

(where "χ is a character" means that χ is linear and multiplicative, with $\chi(1) = 1$). Clearly, a character $\chi \in \mathbb{X}(\mathrm{Sym})$ can be defined by prescribing (at will!) what is the sequence of complex numbers $\chi(e_n)$, $n \geq 1$.

For this talk it is important that Sym carries in fact a bialgebra structure, i.e. it has a *counit* $\varepsilon : \mathrm{Sym} \to \mathbb{C}$ and a *comultiplication* $\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}$. These are unital algebra homomorphisms, uniquely determined by the requirements that $\varepsilon(e_n) = 0$ for all $n \geq 1$ and that

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}, \quad \forall\, n \geq 0 \ \ \text{(with convention } e_0 := 1).$$

With these added operations (and due to the nice properties that they turn out to have) Sym becomes what is called a *graded connected Hopf algebra.*

As a consequence of the bialgebra structure, one gets an operation of convolution $f \star g$ for two linear functionals $f, g : \mathrm{Sym} \to \mathbb{C}$. In particular one can convolve characters, and it is easy to see that the space of characters $\mathbb{X}(\mathrm{Sym})$ becomes a group under convolution.

A.3. The generators y_n for Sym

For the connection between $\boxtimes$ and Sym, we will need another sequence of generators for Sym, which we will denote as $(y_n)_{n=1}^{\infty}$. Every y_n is a homogeneous symmetric function of degree $n-1$. The first few y_n's are like this:

$$y_1 = 1, \quad y_2 = e_1, \quad y_3 = e_2 + e_1^2, \quad y_4 = e_3 + 3e_1e_2 + e_1^3, \ldots$$

The main point about the y_n's is that the comultiplication $\Delta(y_n) \in \mathrm{Sym} \otimes \mathrm{Sym}$ has a special writing in terms of y_m's with $m \leq n$. For the low order cases given above we can calculate by hand:

$$\Delta(y_1) = 1 \otimes 1 = y_1 \otimes y_1, \quad \Delta(y_2) = 1 \otimes e_1 + e_1 \otimes 1 = y_1^2 \otimes y_2 + y_2 \otimes y_1^2,$$

$$\Delta(y_3) = \Delta(e_2) + (\Delta(e_1))^2 = \cdots = y_1^3 \otimes y_3 + 3y_1y_2 \otimes y_1y_2 + y_3 \otimes y_1^3.$$

In order to indicate the formula for y_n and for $\Delta(y_n)$ for general $n \in \mathbb{N}$, we need to introduce a bit of terminology coming from non-crossing partitions.

Some NC(n) terminology.

• For every $n \in \mathbb{N}$, we denote by $NC(n)$ the set of all non-crossing partitions of $\{1, \ldots, n\}$. A partition in $NC(n)$ is thus an object of the form $\pi = \{V_1, \ldots, V_p\}$ with $V_1, \ldots, V_p$ non-empty subsets of $\{1, \ldots, n\}$ such that $V_1 \cup \cdots \cup V_p = \{1, \ldots, n\}$, where for $i \neq j$ we have that $V_i \cap V_j = \emptyset$ and that V_i, V_j do not cross.

• $NC(n)$ has a natural partial order by *reverse refinement*, which makes it become a lattice.

• There exists an important lattice anti-isomorphism $Kr : NC(n) \to NC(n)$, called the *Kreweras complementation map.*

• A useful notation: for every $n \geq 1$ and every $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ we denote

$$e_\pi := e_{|V_1|} \cdots e_{|V_p|} \in \mathrm{Sym}.$$

Now we can return to the y_n's and we can state precisely what is their definition and how they comultiply.

Definition. We put $y_1 = 1$, and for every $n \geq 2$ we put

$$y_n := \sum_{\pi \in \ NC(n-1)} e_\pi.$$

[E.g. $y_3 = e_1^2 + e_2$, a sum of two terms, corresponding to the two partitions of $NC(2)$.]
It is convenient that besides the y_n's themselves we also consider the corresponding family of symmetric functions "y_π" – that is, for $\pi = \{V_1, \ldots, V_p\}$ in $NC(n)$ we will denote

$$y_\pi := y_{|V_1|} \cdots y_{|V_p|} \in \mathrm{Sym}.$$

Remark. The polynomial formulas connecting y_n's to e_n's can be reversed. As a consequence, one can also define characters $\chi \in \mathbb{X}(\mathrm{Sym})$ by prescribing at will what are to be the values $\chi(y_n) \in \mathbb{C}$, $n \geq 2$ (to which we add the condition $\chi(y_1) = \chi(1) = 1$).

Proposition. For every $n \geq 1$ we have

$$\Delta(y_n) = \sum_{\pi \in NC(n)} y_\pi \otimes y_{Kr(\pi)}$$

(where $Kr(\pi) \in NC(n)$ is the Kreweras complement of π).

A.4. The relation between $(\mathcal{G}, \boxtimes)$ and Sym

We can now state the main theorem of part A of the talk. The theorem describes the group $(\mathcal{G}, \boxtimes)$ in terms of symmetric functions.

Definition. Let $\mu \in \mathcal{G}$ be given, and consider the R-transform $R_\mu(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ (where $\alpha_1 = \mu(X) = 1$). We denote by χ_μ the character in $\mathbb{X}(\mathrm{Sym})$ uniquely determined by the requirement that $\chi(y_n) = \alpha_n, \forall\, n \geq 1$.

Theorem. The map $\mathcal{G} \ni \mu \mapsto \chi_\mu \in \mathbb{X}(\mathrm{Sym})$ gives a group isomorphism between $(\mathcal{G}, \boxtimes)$ and the group of characters $(\, \mathbb{X}(\mathrm{Sym}), \star)$.

Idea of Proof. As mentioned at the end of section A.1, one has a summation formula which describes explicitly the coefficients of the R-transform $R_{\mu \boxtimes \nu}$ in terms of the coefficients of R_μ and R_ν. This matches the formula found for $\Delta(y_n)$ in the proposition stated at the end of section A.3, and leads to $\chi_{\mu \boxtimes \nu} = \chi_\mu \star \chi_\nu$.

B. Generalization to joint distributions of k-tuples

In this part of the talk we fix a positive integer k.

B.1. Free multiplicative convolution on $\mathcal{D}_{\mathrm{alg}}(k)$

We consider the algebra $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ of polynomials in non-commuting indeterminates $X_1, \ldots, X_k$, and we denote

$$\mathcal{D}_{\mathrm{alg}}(k) := \{\mu : \mathbb{C}\langle X_1, \ldots, X_k \rangle \to \mathbb{C} \mid \mu \text{ linear}, \ \mu(1) = 1\}.$$

We think of the functionals in $\mathcal{D}_{\mathrm{alg}}(k)$ as of some (algebraic caricatures of) joint distributions for k-tuples of elements in a noncommutative probability space. More precisely, if $(\mathcal{A}, \varphi)$ is a noncommutative probability space and if $a_1, \ldots, a_k \in \mathcal{A}$, then the *distribution* of the k-tuple $(a_1, \ldots, a_k)$ (with respect to φ) is the functional $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$ uniquely determined by the requirement that

$$\mu(X_{i_1} \cdots X_{i_n}) = \varphi(a_{i_1} \cdots a_{i_n}), \quad \forall\, n \in \mathbb{N}, \ 1 \leq i_1, \ldots, i_n \leq k.$$

The main role in part B of the talk is played by a binary operation on $\mathcal{D}_{\mathrm{alg}}(k)$, still denoted as $\boxtimes$ and called “free multiplicative convolution”, which follows now the multiplication of free k-tuples of elements in a noncommutative probability space. The precise statement for what this means is that, for every $\mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$, the following happens:

$$\left\{ \begin{array}{l} \text{if } (\mathcal{A}, \varphi) \text{ is a noncommutative probability space} \\ \quad \text{and if } a_1, \ldots, a_k, b_1, \ldots, b_k \text{ are in } \mathcal{A} \\ \quad \text{and if } \{a_1, \ldots, a_k\} \text{ is free from } \{b_1, \ldots, b_k\} \\ \quad \text{and if } \{a_1, \ldots, a_k\} \text{ has distribution } \mu \text{ while } \{b_1, \ldots, b_k\} \text{ has distribution } \nu \\ \text{then } (a_1 b_1, \ldots, a_k b_k) \text{ has distribution } \mu \boxtimes \nu. \end{array} \right.$$

Same as in the 1-variable case, the rules of free independence allow us to write concrete formulas for how $\mu \boxtimes \nu$ acts on monomials, in terms of how μ and ν act on monomials. For instance for every $1 \leq i \leq k$ we get

$$(\mu \boxtimes \nu)(X_i) = \mu(X_i) \cdot \nu(X_i),$$

and for every $1 \leq i, j \leq k$ we get

$$(\mu \boxtimes \nu)(X_i X_j) = \mu(X_i)\mu(X_j)\nu(X_i X_j) + \mu(X_i X_j)\nu(X_i)\nu(X_j) - \mu(X_i)\mu(X_j)\nu(X_i)\nu(X_j).$$

Verification for the latter: we take free k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ with distributions μ and ν, and calculate

$$\begin{aligned}(\mu \boxtimes \nu)(X_i X_j) &= \varphi(\,(a_i b_i)\,(a_j b_j)\,) \\ &= \varphi(a_i)\varphi(a_j)\varphi(b_i b_j) + \varphi(a_i a_j)\varphi(b_i)\varphi(b_j) - \varphi(a_i)\varphi(a_j)\varphi(b_i)\varphi(b_j) \\ &= \mu(X_i)\mu(X_j)\nu(X_i X_j) + \mu(X_i X_j)\nu(X_i)\nu(X_j) - \mu(X_i)\mu(X_j)\nu(X_i)\nu(X_j).\end{aligned}$$

B.2. Hopf algebra approach to $\boxtimes$ in k variables

In the same vein as in the 1-variable case, let us put

$$\mathcal{G}_k := \{\mu \in \mathcal{D}_{\text{alg}}(k) \mid \mu(X_i) = 1, \ \forall\, 1 \leq i \leq k\}.$$

It is easy to verify that $(\mathcal{G}_k, \boxtimes)$ is a group (which is not commutative, if $k \geq 2$).

Natural Question: Identify what is the group $(\mathcal{G}_k, \boxtimes)$.

The theorem presented at the end of part A of the talk (which would correspond to the case when $k = 1$) points to a Hopf algebra approach to this question. Indeed, the said theorem comes to an isomorphism $(\mathcal{G}_1, \boxtimes) \simeq (\mathbb{X}(\text{Sym}), \star)$ obtained by using the R-transform. This should then extend to arbitrary $(\mathcal{G}_k, \boxtimes)$, as the R-transform arguments can be easily generalized to several variables! The only additional thing needed here is to invent a suitable Hopf algebra which takes over the role of Sym. We will denote this Hopf algebra by $\mathcal{Y}_k$. It is, simply, a commutative algebra of polynomials:

$$\mathcal{Y}_k = \mathbb{C}[\, y_w \mid w \in \mathcal{W}_k], \quad \text{where } \mathcal{W}_k = \cup_{\ell=2}^{\infty} \{1, \ldots, k\}^{\ell}$$

(the elements of $\mathcal{W}_k$ are "words" over the alphabet $\{1, \ldots, k\}$). What will relate $\mathcal{Y}_k$ to $\boxtimes$ is its comultiplication, which goes like this:

• we first introduce (in the natural way) elements $y_{w;\pi} \in \mathcal{Y}_k$, defined for every $\pi \in NC(n)$ and every $w \in \mathcal{W}_k$ such that $|w| = n$;

• we then *define* the comultiplication Δ of $\mathcal{Y}_k$ via the requirement that

$$\Delta(y_w) = \sum_{\pi \in NC(n)} y_{w;\pi} \otimes y_{w;Kr(\pi)}, \text{ for } w \in \mathcal{W}_k \text{ with } |w| = n.$$

Once the bialgebra $\mathcal{Y}_k$ is in place, we can repeat the same development which led to the theorem from section A.4 of the talk.

Definition. Let $\mu \in \mathcal{G}_k$ be given, and consider the R-transform

$$R_\mu(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \; + \sum_{w \in \mathcal{W}_k} \alpha_w z_w,$$

where we used the shorthand notation $z_w := z_{i_1} \cdots z_{i_n}$ for $w = (i_1, \ldots, i_n) \in \mathcal{W}_k$. We denote by χ_μ the character in $\mathbb{X}(\mathcal{Y}_k)$ which is uniquely determined by the requirement that $\chi(y_w) = \alpha_w$, $\forall\, w \in \mathcal{W}_k$.

Theorem. The map $\mathcal{G}_k \ni \mu \mapsto \chi_\mu \in \mathbb{X}(\mathcal{Y}_k)$ gives a group isomorphism between $(\mathcal{G}_k, \boxtimes)$ and the group of characters $(\,\mathbb{X}(\mathcal{Y}_k), \star)$.

Idea of Proof. As mentioned at the end of section A.1, one has a summation formula which describes explicitly the coefficients of the R-transform $R_{\mu \boxtimes \nu}$ in terms of the coefficients of R_μ and R_ν. This matches the formula used to define $\Delta(y_w)$ and gives $\chi_{\mu \boxtimes \nu} = \chi_\mu \star \chi_\nu$.

B.3. Partial linearization of $\boxtimes$ on $\mathcal{G}_k$, by LS-transform

Remark. Every $\xi \in \mathbb{X}(\mathcal{Y}_k)$ has a logarithm $\log(\xi)$. This is the linear functional $\eta : \mathcal{Y}_k \to \mathbb{C}$ defined as

$$\eta = -\sum_{j=1}^{\infty} \frac{1}{j} (\varepsilon - \xi)^{\star j}$$

(the infinite sum makes sense due to a natural grading that one has on $\mathcal{Y}_k$, which reduces all calculations to finite sums). Moreover, if $\xi_1, \xi_2 \in \mathbb{X}(\mathcal{Y}_k)$ are such that $\xi_1 \star \xi_2 = \xi_2 \star \xi_1$, then $\log(\xi_1 \star \xi_2) = \log(\xi_1) + \log(\xi_2)$.

Definition. Let $\mu \in \mathcal{G}_k$ be given, and let $\chi_\mu \in \mathbb{X}(\mathcal{Y}_k)$ be defined as in section B.2 of the talk. The power series

$$LS_\mu(z_1, \ldots, z_k) := \sum_{w \in \mathcal{W}_k} \Big((\log \chi_\mu(y_w)) \Big) z_w$$

is called the *LS-transform of μ.*

Remark. One also has an alternate combinatorial formula for the values $\log \chi_\mu(y_w)$ (i.e. for the coefficients of LS_μ). This formula expresses directly the values $\log \chi_\mu(y_w)$ in terms of the coefficients of R_μ, by using summations over chains in the lattices $NC(n)$. It is rather stuffy, but has the merit that it makes clear the following fact: given an arbitrary collection of complex numbers $\{\gamma_w \mid w \in \mathcal{W}_k\}$, there exists a unique distribution $\mu \in \mathcal{D}_{\text{alg}}(k)$ such that

$$LS_\mu(z_1, \ldots, z_k) = \sum_{w \in \mathcal{W}_k} \gamma_w z_w.$$

Indeed: by starting from the given $\{\gamma_w \mid w \in \mathcal{W}_k\}$ we can calculate backwards in the above mentioned summations over chains in $NC(n)$, and we obtain another collection of complex numbers $\{\alpha_w \mid w \in \mathcal{W}_k\}$, with the following property:

$$\left\{ \begin{array}{l} \text{A distribution } \mu \in \mathcal{D}_{\text{alg}}(k) \text{ has } LS_\mu(z_1, \ldots, z_k) = \sum_{w \in \mathcal{W}_k} \gamma_w z_w \\ \text{ if and only if it has } R_\mu(z_1, \ldots, z_k) = \sum_{i=1}^k z_i + \sum_{w \in \mathcal{W}_k} \alpha_w z_w. \end{array} \right.$$

This entails the existence and uniqueness of μ with LS-transform equal to $\sum_{w \in \mathcal{W}_k} \gamma_w z_w$, due to the known fact that a distribution in $\mathcal{D}_{\text{alg}}(k)$ can be defined by prescribing (at will) what is its R-transform.

The partial linearization of $\boxtimes$ on $\mathcal{G}_k$ is then stated as follows.

Theorem. $LS_{\mu \boxtimes \nu} = LS_\mu + LS_\nu$, for any $\mu, \nu \in \mathcal{G}_k$ such that $\mu \boxtimes \nu = \nu \boxtimes \mu$.

Proof – immediate! (From $\mu \boxtimes \nu = \nu \boxtimes \mu$ we get that $\chi_\mu \star \chi_\nu = \chi_{\mu \boxtimes \nu} = \chi_\nu \star \chi_\mu$, hence that $\log \chi_{\mu \boxtimes \nu} = \log \chi_\mu + \log \chi_\nu$, and the result follows from the definition of the LS-transform.)

The name "LS-transform" is meant to be suggestive of "log of the S-transform"; this is justified by the following result.

Theorem. Suppose that $k = 1$, hence $\mathcal{G}_k = \mathcal{G}_1 = \mathcal{G}$. For $\mu \in \mathcal{G}_1$ one can then consider both the S-transform $S_\mu(z)$ (as reviewed in part A of the talk) and the LS-transform $LS_\mu(z)$. These two power series are related by the formula

$$LS_\mu(z) = -z \log S_\mu(z).$$

Proof – not immediate! (One needs to use the connection between R_μ and the series $1/S_\mu$, and compare it to the connection in Sym between e_n's and power sum symmetric functions.)

C. A problem: $\boxtimes$-convolution powers for k-tuples?

In this part of the talk we use the same positive integer k and the same framework $(\mathcal{D}_{\text{alg}}(k), \mathcal{G}_k, \mathcal{Y}_k, \ldots)$ as in part B above.

C.1. $\boxtimes$-convolution powers in $\mathcal{G}_k$

Remark. The operation $\boxtimes$ is associative on $\mathcal{D}_{\text{alg}}(k)$, so for $\mu \in \mathcal{D}_{\text{alg}}(k)$ and $p \in \mathbb{N}$ it makes sense (and is quite natural) to denote

$$\mu^{\boxtimes p} := \mu \boxtimes \mu \boxtimes \cdots \boxtimes \mu, \quad \text{with } p \text{ occurrences of } \mu \text{ on the right-hand side.}$$

It is also clear that for $\mu \in \mathcal{D}_{\text{alg}}(k)$ and $p, q \in \mathbb{N}$ we have

$$\left(\mu^{\boxtimes p} \right) \ \boxtimes \ \left(\mu^{\boxtimes q} \right) = \mu^{\boxtimes (p+q)} = \left(\mu^{\boxtimes q} \right) \ \boxtimes \ \left(\mu^{\boxtimes p} \right).$$

So if we assume $\mu \in \mathcal{G}_k$, then the first theorem from section B.3 of the talk can be applied to infer that:

$$LS_{\mu^{\boxtimes p}} = p \cdot LS_\mu, \quad \forall\, p \in \mathbb{N}.$$

Based on this formula, we can introduce the following more general concept, where the exponent p is no longer constrained to be an integer.

Definition. Let $\mu \in \mathcal{G}_k$ and $p > 0$ be given. The convolution power $\mu^{\boxtimes p}$ is the distribution $\nu \in \mathcal{G}_k$ which is uniquely determined by the requirement that $LS_\nu = p \cdot LS_\mu$.

C.2. $\boxtimes$-convolution powers in $\mathcal{G}_k^{(*)}$

The $\boxtimes$-convolution powers are interesting to study when one considers distributions μ living in a C^*-algebraic framework. So instead of noncommutative probability spaces as above (where "$\mathcal{A}$" was allowed to be any unital algebra over $\mathbb{C}$) we want to look at C^*-*probability spaces* $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital C^*-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a *positive* linear functional normalized such that $\varphi(1) = 1$.

Notation. We denote by $\mathcal{G}_k^{(*)}$ the set of distributions $\mu \in \mathcal{G}_k$ which arise from k-tuples of selfadjoint elements in C^*-framework. That is: a distribution $\mu \in \mathcal{G}_k$ belongs to $\mathcal{G}_k^{(*)}$ if and only if there exist a C^*-probability space $(\mathcal{A}, \varphi)$ and selfadjoint elements $a_1, \ldots, a_k \in \mathcal{A}$ such that

$$\mu(X_{i_1} \cdots X_{i_n}) = \varphi(a_{i_1} \cdots a_{i_n}), \quad \forall\, n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq k.$$

Remark. The subset $\mathcal{G}_k^{(*)}$ of $\mathcal{G}_k$ can also be approached without referring explicitly to C^*-probability spaces – the essential ingredient of this other approach is a positivity condition placed on a distribution μ.

More precisely, let us endow $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ with the $*$-operation uniquely determined by the requirement that $X_i = X_i^*$, $1 \leq i \leq k$. (So we have the formula

$$(X_{i_1} \cdots X_{i_n})^* = X_{i_n} \cdots X_{i_1}, \quad \forall\, n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq k,$$

which extends by anti-linearity to define P^* for every $P \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$.) It turns out that a distribution $\mu \in \mathcal{G}_k$ belongs to the smaller set $\mathcal{G}_k^{(*)}$ defined above if and only if it satisfies the following two conditions (i)+(ii):

$$\begin{cases} \text{(i)} \quad \mu(P^* P) \geq 0, \quad \forall\, P \in \mathbb{C}\langle X_1, \ldots, X_k\rangle \\ \\ \text{(ii)} \quad \exists\, r > 0 \text{ such that } |\mu(X_{i_1} \cdots X_{i_n})| \leq r^n \\ \qquad \text{for all } n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq k. \end{cases}$$

The proof that $\big(\mu \in \mathcal{G}_k^{(*)}\big) \Rightarrow \big(\text{(i)+(ii) hold}\big)$ is immediate. For the proof of the converse, one uses a version of the GNS construction.

A comment related to the conditions (i)+(ii): when given $\mu \in \mathcal{G}_k$ for which we want to see if $\mu \in \mathcal{G}_k^{(*)}$, it is usually immediate to check whether (ii) holds or not, but most of the time it is quite a challenge to verify the positivity condition (i). In fact, rather than proving that (i) holds, it may be easier to refer directly to the definition of $\mathcal{G}_k^{(*)}$ which was given above, and to construct an *operator model* for μ (that is, construct an explicit k-tuple of selfadjoint elements in a C^*-probability space which have the given μ as joint distribution).

So now, here is the problem announced in the title of this part of the talk.

A loosely stated problem. *Give interesting examples of μ and p with $\mu \in \mathcal{G}_k^{(*)}$ and $p > 0$, such that the convolution power $\mu^{\boxtimes p}$ (which is defined as an element of the larger set $\mathcal{G}_k$) still belongs to $\mathcal{G}_k^{(*)}$.*

Remark. It is easy to give examples as required above, in the special case when $k = 1$. This is because of the following fact:

$$\left\{ \begin{array}{l} \text{if } \mu \in \mathcal{G}_1^{(*)} \text{ can be represented as the distribution of a \textit{positive element}} \\ \quad \text{in a } C^*\text{-probability space,} \\ \text{then } \mu \boxtimes \nu \in \mathcal{G}_1^{(*)} \text{ for all } \nu \in \mathcal{G}_1^{(*)}. \end{array} \right.$$

Verification of this fact: $\mu \boxtimes \nu$ can be represented as the distribution of ab, with a free from b in a C^*-probability space $(\mathcal{A}, \varphi)$, and where $a \geq 0, b = b^*$. The element ab of $\mathcal{A}$ isn't generally selfadjoint, but has the same distribution as the selfadjoint element $\sqrt{a}\, b \sqrt{a}$, and this gives us that $\mu \boxtimes \nu \in \mathcal{G}_1^{(*)}$.

In particular, we see that $\mu^{\boxtimes p} \in \mathcal{G}_1^{(*)}$ for every $\mu \in \mathcal{G}_1^{(*)}$ described as above (distribution of a positive element in a C^*-probability space) and for every $p \in \mathbb{N}$. For such μ, Belinschi and Bercovici [1] have proved that the conclusion $\mu^{\boxtimes p} \in \mathcal{G}_1^{(*)}$ holds for all $p \in [1, \infty)$, without having to assume that p is integer. The methods from the Belinschi-Bercovici paper are complex analytic, and are specific to the case when $k = 1$.

So then: in the problem given above, the examples that are being asked for should be 'interesting' in a strict multi-variable sense, with $k \geq 2$ and where μ is not coming in some obvious way from 1-variable distributions. In such a strict multi-variable frame it would be interesting to even find examples where p is integer, $p \geq 2$.

References

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