Topics in Quantum Information — **U. Waterloo CO 781, Fall 2006**
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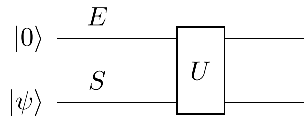
Generalized Measurements and Quantum Operations — **September 9, 2006**

1 General Measurements

Consider the following scenario:



We fix the input state for the environment E to be $|0\rangle$ while the input state $|\psi\rangle$ for the system S is unknown. We are interested in figuring out how the system evolves given that it may interact with the environment. This may be modelled by a unitary U performed on the starting states. Here we shall make the assumption that the environment and the system start off in a product state of dimension $d \times s$. Since we fix the starting state for the environment to $|0\rangle$, the unitary operation may be modelled by an isometry on S given by

$$\begin{aligned} U\,|0\rangle_E\,|\psi\rangle_S &= \sum_k |k\rangle_E \otimes [U_{0k}\,|\psi\rangle]_S \\ \sum_k U_{0k}^\dagger U_{0k} &= I_S \end{aligned}$$

Now, if we measure E in computational basis, then the outcome is k with probability

$$\Pr(k) = \|\, U_{0k}\,|\psi\rangle\,\|^2 = \mathrm{Tr}[U_{0k}^\dagger U_{0k}\,|\psi\rangle\langle\psi|]$$

The state after the measurement is given by

$$\text{Post measurement state } \times \Pr(k) = U_{0k}\,|\psi\rangle\langle\psi|\,U_{0k}^\dagger$$

1.1 POVM Formalism

The POVM (Positive Operator Valued Measure) formalism is used when we do not care about or discard the state of the system after a measurement has been carried out. Now, the probability of obtaining outcome k is given by

$$\Pr(k) = \mathrm{Tr}(M_k \rho)$$

where we have $M_k = U_{0k}^\dagger U_{0k} \geq 0$ (i.e. the M_k are positive operators) and $\sum_k M_k = I_S$. The operators M_k are referred to as the elements of a POVM. The state after measurement conditioned on obtaining ouput k is given by

$$\sqrt{M_k}\rho\sqrt{M_k}$$

2 Quantum Operations

The unitary evolution and measurement process that we have already seen may all be thought of as quantum operations. The above formulations assumed that we had access to the environment and were able to carry out measurements on it to obtain the output k. However, this may not always be possible. Here we shall give a few equivalent formulations of quantum operations. The one we choose to use in a specific scenario depends on the task at hand. In general we want to capture the following process

$$\rho \rightarrow \mathcal{E}(\rho)$$

The mapping $\mathcal{E}(\rho)$ in the above process is referred to as a quantum operation.

Definition 1 *The mapping $\mathcal{E}$ is called positive if it maps positive input to positive output.*

Definition 2 *The mapping $\mathcal{E}$ is called completely positive if $\mathcal{E}$ is positive and $(I \otimes \mathcal{E})$ is also positive.*

Claim 3 *$\mathcal{E}$ is completely positive on a d dimensional space if and only if $(I \otimes \mathcal{E})$ is positive for I acting on a d dimensional space.*

2.1 Operator-sum / Kraus Representation

The Kraus representation may be stated as

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$$

with $\sum_k A_k^\dagger A_k = I$ (note the change of notation $U_{0k} \rightarrow A_k$). Here the operators $\{A_k\}$ are referred to as the Kraus operators.

Question 4 *Let $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger = \sum_l B_l \rho B_l^\dagger$, where $\sum_k A_k^\dagger A_k = \sum_l B_l^\dagger B_l = I$.*
How are A_k and B_l related? Furthermore, given that the dimension of the system is d_S, what upper bound can we obtain on the dimension of the environment d_E?

2.2 Jamiolkowski-Choi Isomorphism

If $\mathcal{E}$ is

1. Completely Positive
2. Linear
3. taking $d \times d$ matrices to $d' \times d'$ matrices

then, the mapping has the form

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$$

In other words, all maps satisfying the three stated properties is of the Kraus form.

Also, the mapping has a unique representation in the following form, the so called "Jamiolkowski-Choi matrix,

$$I_A \otimes \mathcal{E}_B(\Phi).$$

Here Φ is a maximally entangled state on a $d \otimes d$ system, i.e.

$$\Phi = \frac{1}{d} \left(\sum_i |i\rangle_A |i\rangle_B \right) \left(\sum_j \langle j|_B \langle j|_B \right)$$

The analysis for obtaining the A_k's is done in *quant-ph/0201119*.

2.3 Other Representations

Nielsen and Chuang define the χ-representation in *quant-ph/9610001*. Let $\{B_l\}$ be a basis for $d' \times d$ matrices over $\mathbb{C}$. For any

$$
\begin{aligned}
\mathcal{E}(\rho) &= \sum_k A_k \rho A_k^\dagger \\
\text{Let } A_k &= \sum_l c_{lk} B_l
\end{aligned}
$$

Then, the mapping is given by

$$
\mathcal{E}(\rho) = \sum_{ll'} \underbrace{\sum_k c_{lk} c_{l'k}^*}_{\chi_{ll'}} B_l \rho B_{l'}^\dagger .
$$

When $d = d'$, a convenient set of $\{B_l\}$ is the set of all (generalized) Pauli operators.

Question: what are the general properties of χ?