ASSIGNMENT 2 CO 481/CS 467/PHYS 467 (Winter 2014)

Due in class on Tuesday, February 4.

1. *Distinguishing states with local operations and classical communication.*

   Suppose Alice and Bob are each given one qubit of a two-qubit state that is promised to be either $|\psi\rangle$ or $|\phi\rangle$, for some fixed states $|\psi\rangle, |\phi\rangle$. Working together, their goal is to tell which state they have using only local measurements. They can each apply some one-qubit gate to their part of the state and then measure in the computational basis. They can then compare their measurement results by exchanging classical information, but they cannot send qubits or perform any two-qubit gates. For each pair of states, either give such a procedure that Alice and Bob can use to distinguish the states perfectly, or explain why this is not possible.

   (a) [1 point] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

   (b) [1 point] $|\psi\rangle = |00\rangle$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

   (c) [3 points] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

2. *Entanglement swapping.* Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement, but fortunately, they have a mutual friend, Charlie. Alice shares a copy of $|\beta_{00}\rangle$ with Charlie, and Bob also shares a copy of $|\beta_{00}\rangle$ with Charlie.

   (a) [1 point] Write the initial state using Dirac notation in the computational basis, with the first qubit belonging to Alice, the second and third qubits belonging to Charlie, and the fourth qubit belonging to Bob.

   (b) [5 points] Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.

   (c) [2 points] Describe a protocol whereby Charlie sends a classical message to Alice, and Alice processes her quantum state, such that Alice and Bob share the state $|\beta_{00}\rangle$ at the end of the protocol.

3. *The Hadamard gate and qubit rotations*

   (a) [3 points] Suppose that $(n_x, n_y, n_z) \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$. Show that

   $$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i \sin(\tfrac{\theta}{2})\, (n_x X + n_y Y + n_z Z).$$

   (b) [2 points] Find a unit vector $(n_x, n_y, n_z) \in \mathbb{R}^3$ and numbers $\phi, \theta \in \mathbb{R}$ so that

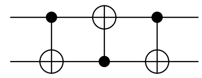
   $$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

   where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

   (c) [3 points] Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

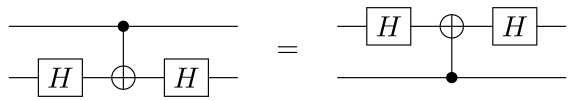
4. *Circuit identities.*

(a) [2 points] Show that the following circuit swaps two qubits:

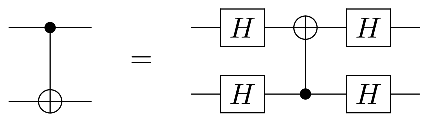


(b) [1 point] Verify that $HXH = Z$.

(c) [3 points] Verify the following circuit identity:



(d) [2 points] Verify the following circuit identity:



Give an interpretation of this identity.

5. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

(a) [1 point] $\{H, T\}$

(b) [2 points] $\{\textsc{cnot}, T\}$

(c) [2 points] $\{\textsc{cnot}, H\}$

(d) [3 points] $\{\textsc{c}Z, K, T\}$, where $\textsc{c}Z$ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$

(e) [challenge problem] $\{\textsc{cnot}, H, T^2\}$

(f) [challenge problem] $\{\textsc{c}T^2, H\}$, where $\textsc{c}T^2$ denotes a controlled-T^2 gate

6. *Asymptotic notation.* Indicate whether the following statements are true or false.

(a) [1 point] $10n^3 + 3n^2 \in O(n^4)$

(b) [1 point] $10n^3 + 3n^2 \in \Omega(n^4)$

(c) [1 point] $1/n^2 \in O(1/n)$

(d) [1 point] $(2^n)^2 \in O(2^n)$

(e) [1 point] $(2^n)^2 \in 2^{\Theta(n)}$