ASSIGNMENT 5 CO 481/CS 467/PHYS 467 (Winter 2012)

Due in class on Thursday, March 22.

1. *Density matrices.* Consider the ensemble in which the state $|0\rangle$ occurs with probability 1/3 and the state $(|0\rangle - |1\rangle)/\sqrt{2}$ occurs with probability 2/3.

   (a) [2 points] What is the density matrix ρ of this ensemble?

   (b) [2 points] Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.

   (c) [3 points] Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{Tr}(\rho|0\rangle\langle 0|)$, and show that the results are consistent.

   (d) [3 points] How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

2. *Local operations and the partial trace.*

   (a) [4 points] Let $|\psi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$ be a two-qubit state. Let ρ denote the density matrix of $|\psi\rangle$ and let ρ' denote the density matrix of $(I \otimes H)|\psi\rangle$. Compute ρ and ρ'.

   (b) [3 points] Compute $\mathrm{Tr}_B(\rho)$ and $\mathrm{Tr}_B(\rho')$, where B refers to the second qubit.

   (c) [4 points] Let ρ be a density matrix for a quantum system with a bipartite state space $A \otimes B$. Let I denote the identity operation on system A, and let U be a unitary operation on system B. Prove that $\mathrm{Tr}_B(\rho) = \mathrm{Tr}_B((I \otimes U)\rho(I \otimes U^\dagger))$.

   (d) [3 bonus points] Show that the converse of part (c) holds for pure states. In other words, show that if $|\psi\rangle$ and $|\phi\rangle$ are bipartite pure states, and $\mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$, then there is a unitary operation U acting on system B such that $|\phi\rangle = (I \otimes U)|\psi\rangle$.

   (e) [2 bonus points] Does the converse of part (c) hold for general density matrices? Prove or disprove it.

3. *The five-qubit code.* Consider a quantum error correcting code that encodes one logical qubit into five physical qubits, with the logical basis states

$$
\begin{aligned}
|0_L\rangle = \tfrac{1}{4}(&|00000\rangle \\
&+|10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&-|11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\
&-|01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle) \\
|1_L\rangle = \tfrac{1}{4}(&|11111\rangle \\
&+|01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&-|00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\
&-|10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle).
\end{aligned}
$$

   (a) [4 points] Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue +1) of the operators given in equation 10.5.18 of KLM. (Hint: You can show this without explicitly checking every case.)

(b) [5 points] Show that this code can correct an X or Z error acting on any of the five qubits. You should explain how the different possible errors would be reflected by a measurement of the error syndrome.

(c) [1 point] Explain why this means that the code can correct any single-qubit error.

(d) [2 points] Find logical Pauli operators X_L and Z_L such that $X_L|0_L\rangle = |1_L\rangle$, $X_L|1_L\rangle = |0_L\rangle$, $Z_L|0_L\rangle = |0_L\rangle$, and $Z_L|1_L\rangle = -|1_L\rangle$.

(e) [3 bonus points] Give a quantum circuit that computes the syndrome of the five-qubit code.

4. *An error-detecting code.* Suppose we encode one logical qubit into four physical qubits, using the logical basis states

$$|0_L\rangle = \frac{1}{2}(|0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle)$$
$$|1_L\rangle = \frac{1}{2}(|0110\rangle + |1010\rangle + |0101\rangle + |1001\rangle).$$

(a) [3 points] Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue $+1$) of the operators $X \otimes X \otimes I \otimes I$, $I \otimes I \otimes X \otimes X$, and $Z \otimes Z \otimes Z \otimes Z$.

(b) [3 points] Find two distinct single-qubit Pauli errors with the same syndrome. This shows that the code fails to correct an arbitrary single-qubit error.

(c) [3 points] Show that any single-qubit Pauli error anticommutes with at least one of the three operators from part (a). (We say that A and B anticommute if $AB = -BA$.) Explain why this means that the code can detect whether a single-qubit error occurred, even though (by the previous part) it cannot determine how to correct the error.