ASSIGNMENT 2 CO 481/CS 467/PHYS 467 (Winter 2012)

Due in class on Thursday, January 26.

1. *Distinguishing quantum states.* [6 points] Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$

(but does not tell you which). Describe a procedure for guessing which state you were given, succeeding with as high a probability as possible. Express your procedure as a unitary operation followed by a measurement in the computational basis. Also indicate the success probability of your procedure. (You do not need to prove that your procedure is optimal, but your grade will depend on how close your procedure is to optimal.)

2. *Entanglement swapping.* Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement, but fortunately, they have a mutual friend, Charlie. Alice shares a copy of $|\beta_{00}\rangle$ with Charlie, and Bob also shares a copy of $|\beta_{00}\rangle$ with Charlie.

   (a) [1 point] Write the initial state using Dirac notation in the computational basis, with the first qubit belonging to Alice, the second and third qubits belonging to Charlie, and the fourth qubit belonging to Bob.

   (b) [5 points] Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.

   (c) [2 points] Describe a protocol whereby Charlie sends a classical message to Alice, and Alice processes her quantum state, such that Alice and Bob share the state $|\beta_{00}\rangle$ at the end of the protocol.

3. *The Hadamard gate and qubit rotations*

   (a) [3 points] Suppose that $(n_x, n_y, n_z) \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i\sin(\tfrac{\theta}{2})\,(n_x X + n_y Y + n_z Z).$$

   (b) [2 points] Find a unit vector $(n_x, n_y, n_z) \in \mathbb{R}^3$ and numbers $\phi, \theta \in \mathbb{R}$ so that

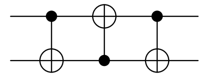
$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

   where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

   (c) [3 points] Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

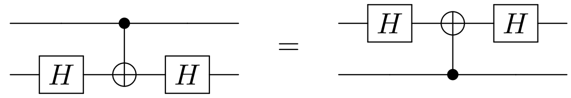
4. *Circuit identities.*

(a) [2 points] Show that the following circuit swaps two qubits:

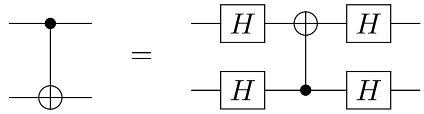


(b) [1 point] Verify that $HXH = Z$.

(c) [3 points] Verify the following circuit identity:



(d) [3 points] Verify the following circuit identity:



Give an interpretation of this identity.

5. *Teleporting through a Hadamard gate.*

(a) [1 point] Write the state $(I \otimes H)|\beta_{00}\rangle$ in the computational basis.

(b) [3 points] Suppose Alice has a qubit in the state $|\psi\rangle$ and also, Alice and Bob share a copy of the state $(I \otimes H)|\beta_{00}\rangle$. If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

(c) [2 points] Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state $H|\psi\rangle$?

6. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

(a) [1 point] $\{H, T\}$

(b) [2 points] $\{\textsc{cnot}, T\}$

(c) [2 points] $\{\textsc{cnot}, H\}$

(d) [3 points] $\{\textsc{c}Z, K, T\}$, where $\textsc{c}Z$ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$

(e) [challenge problem] $\{\textsc{cnot}, H, T^2\}$

(f) [challenge problem] $\{\textsc{c}T^2, H\}$, where $\textsc{c}T^2$ denotes a controlled-T^2 gate