ASSIGNMENT 2 CO 481/CS 467/PHYS 467 (Winter 2010)

Due in class on Wednesday, February 3.

1. *The Hadamard gate and qubit rotations*

   (a) [3 points] Suppose that $(n_x, n_y, n_z) \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$. Show that

   $$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i \sin(\tfrac{\theta}{2})\, (n_x X + n_y Y + n_z Z).$$

   (b) [2 points] Find a unit vector $(n_x, n_y, n_z) \in \mathbb{R}^3$ and numbers $\phi, \theta \in \mathbb{R}$ so that

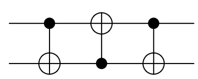
   $$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

   where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

   (c) [3 points] Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

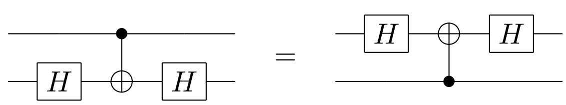
2. *Circuit identities.*

   (a) [2 points] Show that the following circuit swaps two qubits:

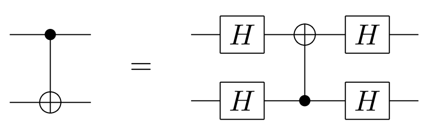
   

   (b) [1 point] Verify that $HXH = Z$.

   (c) [3 points] Verify the following circuit identity:

   (d) [3 points] Verify the following circuit identity:

   Give an interpretation of this identity.

3. *Teleporting through a Hadamard gate.*

   (a) [1 point] Write the state $(I \otimes H)|\beta_{00}\rangle$ in the computational basis.

   (b) [3 points] Suppose Alice has a qubit in the state $|\psi\rangle$ and also, Alice and Bob share a copy of the state $(I \otimes H)|\beta_{00}\rangle$. If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

   (c) [2 points] Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state $H|\psi\rangle$?

4. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

   (a) [1 point] $\{H, T\}$

   (b) [2 points] $\{\textsc{cnot}, T\}$

   (c) [2 points] $\{\textsc{cnot}, H\}$

   (d) [3 points] $\{\textsc{c}Z, K, T\}$, where $\textsc{c}Z$ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$

   (e) [challenge problem] $\{\textsc{c}T^2, H\}$, where $\textsc{c}T^2$ denotes a controlled-T^2 gate

   (f) [challenge problem] $\{\textsc{cnot}, H, T^2\}$

5. *Asymptotic notation.* Indicate whether the following statements are true or false.

   (a) [1 point] $10n^3 + 3n^2 \in O(n^4)$

   (b) [1 point] $10n^3 + 3n^2 \in \Omega(n^4)$

   (c) [1 point] $1/n^2 \in O(1/n)$

   (d) [1 point] $(2^n)^2 \in O(2^n)$

   (e) [1 point] $(2^n)^2 \in 2^{\Theta(n)}$

6. *One-out-of-four search.* Let $f : \{0,1\}^2 \to \{0,1\}$ be a black-box function taking the value 1 on exactly one input. The goal of the one-out-of-four search problem is to find the unique $(x_1, x_2) \in \{0,1\}^2$ such that $f(x_1, x_2) = 1$.

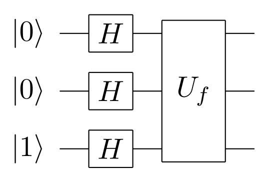
   (a) [1 point] Write the truth tables of the four possible functions f.

   (b) [2 points] How many classical queries are needed to solve one-out-of-four search?

   (c) [4 points] Suppose f is given as a quantum black box U_f acting as

   $$|x_1, x_2, y\rangle \overset{U_f}{\mapsto} |x_1, x_2, y \oplus f(x_1, x_2)\rangle.$$

   Determine the output of the following quantum circuit for each of the possible black-box functions f:

   (d) [2 points] Show that the four possible outputs obtained in the previous part are pairwise orthogonal. What can you conclude about the quantum query complexity of one-out-of-four search?