ASSIGNMENT 1 CO 481/CS 467/PHYS 467 (Winter 2010)

Due in class on Wednesday, January 20.

1. *Universality of reversible logic gates.*

   (a) [3 points] The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1. Show how to implement a CCCNOT gate using Toffoli gates. You may use additional workspace as needed. You may assume that bits in the workspace start with a particular value, either 0 or 1, provided you return them to that value. For a bonus point, give a circuit that works regardless of the values of any bits of workspace.

   (b) [5 points] Show that a Toffoli gate cannot be implemented using any number of CNOT gates, with any amount of workspace. Hence the CNOT gate alone is not universal. (Hint: It may be helpful to think of the gates as performing arithmetic operations on integers mod 2.)

2. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

   (a) [2 points] $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$

   (b) [2 points] $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$

   (c) [2 points] $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

3. *Pauli operators.*

   (a) [1 point] Express each of the three Pauli operators,

   $$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   using Dirac notation in the computational basis.

   (b) [3 points] Find the eigenvalues and the corresponding eigenvectors of each Pauli operator. Express the eigenvectors using Dirac notation.

   (c) [2 points] Write the operator $X \otimes Z$ as a matrix and using Dirac notation (in both cases using the computational basis).

   (d) [2 points] What are the eigenspaces of the operator $X \otimes Z$? Express them using Dirac notation.

4. *Unitary operations and measurements.* Consider the state

   $$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$

   (a) [1 point] Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.

   (b) [3 points] Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?

   (c) [3 points] Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?

(d) [2 points] Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

5. *Distinguishing quantum states.* [6 points] Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$

(but does not tell you which). Describe a procedure for guessing which state you were given, succeeding with as high a probability as possible. Express your procedure as a unitary operation followed by a measurement in the computational basis. Also indicate the success probability of your procedure. (You do not need to prove that your procedure is optimal, but your grade will depend on how close your procedure is to optimal.)

6. *Entanglement swapping.* Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement, but fortunately, they have a mutual friend, Charlie. Alice shares a copy of $|\beta_{00}\rangle$ with Charlie, and Bob also shares a copy of $|\beta_{00}\rangle$ with Charlie.

   (a) [1 point] Write the initial state using Dirac notation in the computational basis, with the first qubit belonging to Alice, the second and third qubits belonging to Charlie, and the fourth qubit belonging to Bob.

   (b) [5 points] Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.

   (c) [2 points] Describe a protocol whereby Charlie sends a classical message to Alice, and Alice processes her quantum state, such that Alice and Bob share the state $|\beta_{00}\rangle$ at the end of the protocol.