Final exam review Math 245 (Winter 2009)

This is a list of the major topics we covered in Math 245. Corresponding sections of the textbook appear in brackets.

1. Building vector spaces
   (a) Direct sums [5.2, 5.4]
   (b) Tensor products
2. Determinants
   (a) 2×2 case, inverting 2×2 matrices [4.1]
   (b) Cofactor expansion [4.2]
   (c) Properties: Multilinearity, behavior under elementary row operations [4.2]
   (d) Computing determinants [4.2]
   (e) $\det(AB) = \det(A)\det(B)$, $\det(A^t) = \det(A)$, $\det(A^{-1}) = 1/\det(A)$ [4.3]
   (f) Cramer's rule [4.3]
   (g) Axiomatic definition [4.5], $\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} A_{i,\pi(i)}$
3. Eigenvalues and eigenvectors
   (a) Definitions [5.1]
   (b) Characteristic polynomial [5.1]
   (c) Eigenspaces [5.2]
   (d) Computing eigenvalues and eigenspaces [5.1, 5.2]
   (e) Diagonalizability [5.2]
   (f) Theorem: A linear transformation on a finite-dimensional vector space is diagonalizable iff its characteristic polynomial splits and the multiplicity of each eigenvalue equals the dimension of the corresponding eigenspace. [5.2]
   (g) Diagonalizability and direct sums [5.2]
   (h) Applications: Computing powers of matrices, solving recurrences, solving differential equations [5.2]
   (i) Invariant subspaces, T-cyclic subspaces [5.4]
   (j) Restriction of a linear transformation to an invariant subspace [5.4]
   (k) Cayley-Hamilton theorem [5.4]
   (l) Minimal polynomial [7.3]
   (m) Probability vectors, stochastic matrices, Markov chains [5.3]
   (n) Matrix limits [5.3]
   (o) Theorem: If A is a stochastic, regular, diagonalizable matrix, then $\lim_{m\to\infty} A^m$ exists and has all columns equal to the unique probability vector that is an eigenvector of A with eigenvalue 1. [5.3]
4. Linear transformations on inner product spaces
   (a) Dual spaces [2.6]
   (b) Adjoint of a linear transformation, existence and uniqueness [6.3]
   (c) Adjoint of a matrix [6.3]

(d) Normal operators/matrices [6.4]
(e) Schur's Theorem [6.4]
(f) Theorem: A linear transformation on a finite-dimensional inner product space is normal iff it has an orthonormal basis of eigenvectors. [6.4]
(g) Self-adjoint operators, Hermitian and symmetric matrices [6.4]
(h) Theorem: A linear transformation on a finite-dimensional, real inner product space is self-adjoint iff it has an orthonormal basis of eigenvectors. [6.4]
(i) Unitary and orthogonal transformations [6.5]
(j) Characterizations of unitarity/orthogonality [6.5]
(k) Unitary/orthogonal equivalence [6.5]
(l) The orthogonal complement of a subspace [6.2]
(m) Projection operators, orthogonal projections [6.6]
(n) Spectral theorem [6.6]
(o) Singular value decomposition [6.7]
(p) Fitting a line through the origin or a hyperplane, minimizing $\|Ax\|$ over unit vectors x
(q) Pseudoinverse [6.7]
(r) Fitting a line, minimizing $\|Ax - b\|$ over x [6.3, 6.7]
(s) Quantum Markov chains (not covered on the final exam)

5. The Jordan canonical form
   (a) Definition [7.1]
   (b) Generalized eigenvectors, generalized eigenspaces [7.1]
   (c) $\mathsf{V} = \bigoplus \mathsf{K}_\lambda$ [7.1]
   (d) Chains of generalized eigenvectors [7.1]
   (e) Theorem: If V is a finite-dimensional vector space and the characteristic polynomial of $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ splits, then T has a Jordan canonical form. [7.1]
   (f) Computing a Jordan canonical form [7.1]

Only major theorems are listed above; the list does not include everything we proved along the way. For the final exam, you should try to be familiar with every definition and every proof we covered, both in class and on the assignments. But in particular, you will be asked to prove at least one of the following:

- For any $A, B \in \mathsf{M}_{n\times n}(\mathbb{F})$, $\det(AB) = \det(A)\det(B)$. [Theorem 4.7]
- If T is a linear transformation with eigenvectors $v_1, \ldots, v_k$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then the vectors $v_1, \ldots, v_k$ are linearly independent. [Theorem 5.5]
- Gerschgorin circle theorem. [Theorem 5.16]
- If V is a finite-dimensional vector space, $\mathsf{T} \in \mathcal{L}(\mathsf{V})$, and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T. [Theorem 5.21]
- A linear transformation T is an orthogonal projection iff it has an adjoint T^* and $\mathsf{T} = \mathsf{T}^2 = \mathsf{T}^*$. [Theorem 6.24]
- If V is a vector space and λ is an eigenvalue of $\mathsf{T} \in \mathcal{L}(\mathsf{V})$, then the set $K_\lambda := \{x \in \mathsf{V} : (T - \lambda\mathsf{I})^p(x) = 0 \text{ for some positive integer } p\}$ is a T-invariant subspace of V. [Theorem 7.1a]