**Comparison of direct sums & tensor products** Math 245 (Winter 2009)

| $\bigoplus$ | $\bigotimes$ |
|---|---|
| With appropriate definitions of addition and scalar multiplication, | |
| $\bigoplus_{i=1}^{n} \mathsf{V}_i \cong \{v_1 \times \cdots \times v_n : v_i \in \mathsf{V}_i\}$ | $\bigotimes_{i=1}^{n} \mathsf{V}_i \cong \operatorname{span}\{v_1 \times \cdots \times v_n : v_i \in \mathsf{V}_i\}$ |
| Vectors written as $v_1 + \cdots + v_n$ (internal)<br>or $(v_1, \ldots, v_n)$ (external)<br>or $v_1 \oplus \cdots \oplus v_n$ (either)<br>where $v_i \in \mathsf{V}_i$. | Vectors written as $a(v_1 \otimes \ldots \otimes v_n)$<br>$+ b(w_1 \otimes \ldots \otimes w_n)$<br>$+ \cdots$<br>where $a, b, \ldots \in \mathbb{F}$, $v_i, w_i, \ldots \in \mathsf{V}_i$. |
| $(v_1 \oplus v_2) + (w_1 \oplus w_2) = (v_1 + w_1) \oplus (v_2 + w_2)$ | $(v_1 \otimes v_2) + (w_1 \otimes v_2) = (v_1 + w_1) \otimes v_2$, and<br>$(v_1 \otimes v_2) + (v_1 \otimes w_2) = v_1 \otimes (v_2 + w_2)$,<br>but $(v_1 \otimes v_2) + (w_1 \otimes w_2)$ sometimes cannot be simplified further |
| $a(v_1 \oplus v_2) = (av_1 \oplus av_2)$ | $a(v_1 \otimes v_2) = (av_1) \otimes v_2 = v_1 \otimes (av_2)$ |
| Basis for $\bigoplus_{i=1}^{n} \mathsf{V}_i$:<br>$\{v_1 \oplus 0 \oplus \cdots \oplus 0 : v_1 \in \gamma_1\}$<br>$\cup \{0 \oplus v_2 \oplus 0 \cdots \oplus 0 : v_2 \in \gamma_2\}$<br>$\cup \cdots$<br>$\cup \{0 \oplus \cdots \oplus 0 \oplus v_n : v_n \in \gamma_n\}$<br>where $\gamma_i$ is a basis for $\mathsf{V}_i$. | Basis for $\bigotimes_{i=1}^{n} \mathsf{V}_i$:<br>$\gamma_1 \otimes \cdots \otimes \gamma_n = \{v_1 \otimes \cdots \otimes v_n : v_i \in \gamma_i\}$<br>where $\gamma_i$ is a basis for $\mathsf{V}_i$. |
| $\dim\left(\bigoplus_{i=1}^{n} \mathsf{V}_i\right) = \sum_{i=1}^{n} \dim(\mathsf{V}_i)$ | $\dim\left(\bigotimes_{i=1}^{n} \mathsf{V}_i\right) = \prod_{i=1}^{n} \dim(\mathsf{V}_i)$ |
| Let $\mathsf{T}_1 \in \mathcal{L}(\mathsf{V}_1, \mathsf{W}_1), \mathsf{T}_2 \in \mathcal{L}(\mathsf{V}_2, \mathsf{W}_2)$. | |
| $\mathsf{T}_1 \oplus \mathsf{T}_2 \in \mathcal{L}(\mathsf{V}_1 \oplus \mathsf{V}_2, \mathsf{W}_1 \oplus \mathsf{W}_2)$ with<br>$(\mathsf{T}_1 \oplus \mathsf{T}_2)(v_1 \oplus v_2) = \mathsf{T}_1(v_1) \oplus \mathsf{T}_2(v_2)$ | $\mathsf{T}_1 \otimes \mathsf{T}_2 \in \mathcal{L}(\mathsf{V}_1 \otimes \mathsf{V}_2, \mathsf{W}_1 \otimes \mathsf{W}_2)$ with<br>$(\mathsf{T}_1 \otimes \mathsf{T}_2)(v_1 \otimes v_2) = \mathsf{T}_1(v_1) \otimes \mathsf{T}_2(v_2)$ |
| Let $A \in \mathsf{M}_{m\times n}(\mathbb{F}), B \in \mathsf{M}_{k,\ell}(\mathbb{F})$. | |
| $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathsf{M}_{m+k,n+\ell}(\mathbb{F})$ | $A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$<br>$\in \mathsf{M}_{mk\times n\ell}(\mathbb{F})$ |