# ASSIGNMENT 8 Math 245 (Winter 2009)

Due at the start of class on Wednesday 18 March.

1. Let $\mathsf{V}$ be a finite-dimensional inner product space over $\mathbb{C}$, and let $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ be a normal linear transformation with spectral decompostion $\mathsf{T} = \sum_{j=1}^{k} \lambda_j \mathsf{P}_j$, where the $\mathsf{P}_j$ are orthogonal projections.
   (a) Show that if $f \in \mathbb{C}[x]$ is a polynomial, then $f(\mathsf{T}) = \sum_{j=1}^{k} f(\lambda_j)\mathsf{P}_j$.
   (b) Define $e^{\mathsf{T}} := \sum_{j=0}^{\infty} \mathsf{T}^j/j!$. Show that $e^{\mathsf{T}} = \sum_{j=0}^{k} e^{\lambda_j}\mathsf{P}_j$.
   (c) More generally, we can define other functions of $\mathsf{T}$ in terms of its spectral decomposition. Given $g : \mathbb{C} \to \mathbb{C}$ (not necessarily a polynomial, or even a function with a Taylor series), let $g(\mathsf{T}) := \sum_{j=1}^{k} g(\lambda_j)\mathsf{P}_j$. Show that $\sqrt{\mathsf{T}^*\mathsf{T}} = \sqrt{\mathsf{T}\mathsf{T}^*} = |\mathsf{T}|$, where $|\cdot|$ denotes the modulus function.

2. Let $B \in \mathsf{M}_{3\times 3}(\mathbb{C})$ denote the matrix

$$B := \begin{pmatrix} x & y & y \\ y & x & y \\ y & y & x \end{pmatrix}.$$

   (a) Find orthogonal projection matrices $P_1, \ldots, P_k$ such that $B = \sum_{j=1}^{k} \lambda_j P_j$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $B$.
   (b) Find a matrix $A \in \mathsf{M}_{3\times 3}(\mathbb{C})$ such that $A^2 = B$.

3. Compute the singular value decompositions of the following matrices:

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad C := \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

4. Let $\mathsf{V}, \mathsf{W}$ be finite-dimensional inner product spaces over $\mathbb{C}$. Prove that for any $x \in \mathsf{V} \otimes \mathsf{W}$, there exist orthonormal bases $\{v_1, \ldots, v_{\dim \mathsf{V}}\}$ for $\mathsf{V}$ and $\{w_1, \ldots, w_{\dim \mathsf{W}}\}$ for $\mathsf{W}$ and scalars $a_1, \ldots, a_n \in \mathbb{C}$ (where $n \leq \min(\dim \mathsf{V}, \dim \mathsf{W})$) such that $x = \sum_{j=1}^{n} a_j v_j \otimes w_j$. (This is called a *Schmidt decomposition* of $x$.)

5. Let $A \in \mathsf{M}_{n\times n}(\mathbb{C})$. Let $\sigma(A)$ denote the largest singular value of $A$, and let

$$\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

   (a) Prove that $\max_x \dfrac{\|Ax\|}{\|x\|} = \sigma(A)$, where the maximum is over all nonzero $x \in \mathsf{M}_{n\times 1}(\mathbb{C})$.
   (b) Prove or disprove: $\sigma(A) \leq \rho(A)$, $\sigma(A) \geq \rho(A)$.
   (c) A function $\nu : \mathsf{M}_{n\times n}(\mathbb{C}) \to \mathbb{R}$ is called a *matrix norm* if it satisfies three axioms: $\nu(A) \geq 0$ for all $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, with equality only when $A$ is the zero matrix; $\nu(cA) = |c|\nu(A)$ for all $c \in \mathbb{C}$ and all $A \in \mathsf{M}_{n\times n}(\mathbb{C})$; and $\nu(A+B) \leq \nu(A) + \nu(B)$ for all $A, B \in \mathsf{M}_{n\times n}(\mathbb{C})$. Is $\sigma$ a matrix norm? How about $\rho$?