ASSIGNMENT 6 Math 245 (Winter 2009)

Due at the start of class on Wednesday 4 March.

1. Let V be a vector space, and let S be a subset of V. The *annihilator* of S is

$$S^0 := \{f \in \mathsf{V}^* : f(s) = 0 \text{ for all } s \in S\}.$$

   (a) Prove that S^0 is a subspace of V^*.

   (b) Prove that $S^0 = \operatorname{span}(S)^0$.

   (c) Suppose V is finite-dimensional, and W is a subspace of V. What is $\dim(\mathsf{W}^0)$?

   (d) Suppose W is a subspace of V. Prove that W^* is naturally isomorphic to $\mathsf{V}^*/\mathsf{W}^0$.

2. Let $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$, where V is a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle_{\mathsf{V}}$ and W is a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle_{\mathsf{W}}$. We call $\mathsf{T}^* \in \mathcal{L}(\mathsf{W}, \mathsf{V})$ an *adjoint* of T provided $\langle \mathsf{T}(v), w \rangle_{\mathsf{W}} = \langle v, \mathsf{T}^*(w) \rangle_{\mathsf{V}}$ for all $v \in \mathsf{V}$ and all $w \in \mathsf{W}$. Prove the following:

   (a) T^* exists and is unique.

   (b) If β is an orthonormal basis for V and γ is an orthonormal basis for W, then $[\mathsf{T}^*]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^*$.

   (c) $\operatorname{rank}(\mathsf{T}^*) = \operatorname{rank}(\mathsf{T})$.

   (d) $\langle \mathsf{T}^*(w), v \rangle_{\mathsf{V}} = \langle w, \mathsf{T}(v) \rangle_{\mathsf{W}}$ for all $v \in \mathsf{V}$ and all $w \in \mathsf{W}$.

3. (a) Let V, W be finite-dimensional inner product spaces, and let $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$. Prove that $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$.

   (b) Let $A \in \mathsf{M}_{n\times m}(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Deduce from part (a) that $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$.

   (c) Let $\langle \cdot, \cdot \rangle$ be an inner product over $\mathbb{F}^n$. The *Gram matrix* of a set of vectors $v_1, v_2, \ldots, v_m \in \mathbb{F}^n$ is the matrix $\Gamma \in \mathsf{M}_{m\times m}(\mathbb{F})$ with entries $\Gamma_{jk} = \langle v_j, v_k \rangle$. Prove that $\operatorname{rank}(\Gamma) = \dim(\operatorname{span}\{v_1, v_2, \ldots, v_m\})$.

4. Let V be an inner product space over $\mathbb{C}$.

   (a) Let $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ be normal. Let $p \in \mathbb{C}[x]$ be a polynomial with complex coefficients. Prove that $p(\mathsf{T})$ is normal.

   (b) Let $\mathsf{U} \in \mathcal{L}(\mathsf{V})$ be self-adjoint. Let $q \in \mathbb{R}[x]$ be a polynomial with real coefficients. Prove that $q(\mathsf{U})$ is self-adjoint.

   (c) Does (b) remain true if q has complex coefficients?

5. Let $x, y \in \mathbb{C}$, and let $B \in \mathsf{M}_{3\times 3}(\mathbb{C})$ be the matrix

$$B := \begin{pmatrix} x & y & y \\ y & x & y \\ y & y & x \end{pmatrix}.$$

   (a) Show that B is normal.

   (b) Compute the eigenvalues of B, and find an *orthonormal* basis of eigenvectors.