ASSIGNMENT 5 Math 245 (Winter 2009)

Due at the start of class on Wednesday 25 February.

1. Let $p \in \mathbb{R}^n$ be a probability vector, and let $A, B \in \mathsf{M}_{n\times n}(\mathbb{R})$ and $C \in \mathsf{M}_{m\times m}(\mathbb{R})$ be stochastic matrices.
   (a) Prove that Ap is a probability vector.
   (b) Prove that BA is a stochastic matrix.
   (c) Prove or disprove: $A \oplus C$ is a stochastic matrix.
   (d) Prove or disprove: $A \otimes C$ is a stochastic matrix.

2. A population of fish has three colors: red, green, and blue. Every year, some of the fish change color, as follows. A quarter of the red fish turn into green fish, and another quarter turn into blue fish, with the rest remaining red. The green fish do not like being green; a third of them turn into red fish, and two thirds turn into blue fish. A sixth of the blue fish turn into red fish, and three quarters turn into green fish, with the rest remaining blue.
   (a) Describe this process as a Markov chain governed by a stochastic matrix A.
   (b) Compute the eigenvalues and eigenvectors of A.
   (c) Compute $\lim_{m\to\infty} A^m$. What can you say about the long-term behavior of the population?
   (d) A matrix $D \in \mathsf{M}_{n\times n}(\mathbb{C})$ is called *doubly stochastic* if both D and D^t are stochastic. Prove that if D is doubly stochastic, regular, and diagonalizable, then $\lim_{m\to\infty} D^m p = u/n$ for any probability vector p, where $u \in \mathbb{C}^n$ is the vector with every component equal to 1.

3. You are looking for a needle in a haystack. The haystack contains $n-1$ pieces of hay and one needle. Your strategy is as follows. First, choose a random item from the haystack. If it is not the needle, put it back in the haystack (forgetting where you put it) and repeat the process. If it is the needle, keep it for all future steps.
   (a) Describe this process by a Markov chain with n states, one for each item in the haystack, corresponding to a stochastic matrix H.
   (b) Is H regular?
   (c) Does $\lim_{m\to\infty} H^m$ exist? If so, find it.
   (d) What is the probability of finding the needle after m steps? How large must m be before this probability is at least $1/2$? (You should give an explicit expression, and also say roughly how big you think m should be as compared to n.)

4. (a) Let $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ be diagonalizable, and suppose that $|\lambda| < 1$ for any eigenvalue λ of A. Define
   $$B_m := I + A + A^2 + \cdots + A^{m-1},$$
   where I is the $n \times n$ identity matrix. Prove that
   $$\lim_{m\to\infty} B_m = (I - A)^{-1}.$$
   (b) Let $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ be diagonalizable, and define
   $$C_m := I + 2A + 3A^2 + \cdots + mA^{m-1}.$$
   When does $\lim_{m\to\infty} C_m$ exist? When it exists, what is it?